STRENGTH OF MATERIALS

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A MANUAL FOR STUDENTS OF ENGINEERING

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PREFACE

THE greater part of the matter contained in the following pages is based on the notes of lectures given to the day and evening students at the Manchester Municipal School of Technology during the last five sessions.

The book is intended for the use of those students of engineering who are desirous of obtaining a working knowledge of the fundamental principles involved in problems of machine and structural design. It should be found useful to candidates for the Third and Honours stages of the Examinations of the Board of Education, the examination for the admission of Associate Members to the Institution of Civil Engineers, as well as the examinations in the Engineering Schools of the Universities.

It will be seen that special attention has been paid to the unequal distribution of stress, and to the limits of elasticity in iron and steel. Many of the examples quoted are taken from experimental results obtained by the writer or his students. It is also to be noted that the majority of the proofs given are similar to those used in most of the text-books.

The author desires to acknowledge his indebtedness to the many writers of books and scientific papers to which he has referred in collecting these notes.

W. C. P.

MANCHESTER, 1907.

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INTRODUCTORY

ALL material used by the engineer, whether it forms part of some piece of mechanism or of a fixed structure, has to withstand the application of force. The structural part must be so proportioned by the engineer who initiates its design as to be able to carry the loads which come upon it without injury to itself, and at the same time without the employment of more material than is necessary. It is for the carrying out of this safe and economical design that a knowledge of the "Strength of Materials" is required.

The causes producing the loads which come upon the various parts of a structure, and their magnitudes as depending on outside effects and upon the form of the structure as a whole, will not be discussed in any detail in what follows. That part of the subject is generally dealt with under the title of "Theory of Structures." Here the reader is more particularly concerned with a knowledge of the relations between the ascertained loads and the dimensions and forms as affecting the stresses in structural members.

The subject of "Strength of Materials" naturally divides itself into two parts. The former of these deals with the nature and intensity of the forces which come upon a part, as depending on its form and the loads which it has to carry. The second portion of the subject relates more especially to the effect which these forces have upon the internal structure of the material itself. One is analytical and depends upon mathematical proofs, the other is descriptive and experimental.

This order will be observed in the following chapters, the general laws bearing upon the relations of stress and strain, which apply equally to all materials, being taken first; to be followed by a more detailed discussion of the effect of stress upon particular materials.

STRENGTH OF MATERIALS

CHAPTER I

STRESS, STRAIN, AND ELASTICITY

Stress.—Stress is the force exerted by a portion of material upon that part adjacent to it.

A stress may act normally to a surface, when it will be either a compressive or a tensile stress. In the former of these the tendency is for the two portions of material on opposite sides of the section to be pressed against one another; in the latter case the tendency is towards a separation of the parts. Or, the stress may be tangential to the surface in question, with a tendency to cause the portion on one side of the section to slide upon the other: this is called a shear stress. Again, the stress may be partly normal and partly tangential.

Load.—This term is applied to the total force which acts upon a structural part. A body acted upon by a load is said to be in a state of stress. Thus, in the case of a metal rod which is withstanding a pair of loads or forces pulling away from one another at the opposite ends, the whole of the material between the two ends is said to be in a state of tensile stress, and it is only the mutual adhesion between the individual particles which prevents the metal from being torn asunder at any point.

Intensity of Stress.—Stress is generally defined as being so many units of force acting upon a unit of the area referred to. The load, on the other hand, is defined as so many units of force, irrespective of the extent of the surface upon which it acts. Thus

in the case of the above tension bar, the load might be given as so many tons, and if the extent of the area of the cross-section were so many square inches, the stress would be given as so many tons acting on each square inch, or as so many *tons per square inch.*

The principal units employed for the measurement of stress are as follows:—

In Great Britain.—For the metals and timber the stresses are usually given in *tons per square inch,* and, less frequently, in *pounds per square inch.*

For brickwork, masonry, and concrete, stresses are given in *tons per square foot,* and sometimes in *pounds per square foot.*

In the United States of America the units employed are respectively *pounds per square inch* and *pounds per square foot.*

In the countries using the metric system, *kilogrammes* or *grammes per square centimetre* or *millimetre* are the units.

Strain.—Strain is the deformation brought about by stress. Stress never occurs without changing the shape of the piece of material on which it acts.

For example, in the case of the steel bar before referred to, a tension load or pull is put upon the bar, causing tensile stress throughout its length, with the result that the bar is stretched: this stretch is the *strain.*

The strain may be temporary or permanent, or part of it may be temporary and part permanent. Under the ordinary working loads which are put upon engineering materials the strains accompanying the stresses are relatively of very small magnitude, and are almost always temporary, disappearing on the removal of the loads. When, however, the loading is carried beyond the working limits the strains become larger, and the material only partially recovers itself on the removal of the load, leaving the remainder of the deformation as permanent strain.

Different Kinds of Stress and Strain.

Simple Stresses.—There are three kinds of simple stress, namely: Tension or pulling; compression or thrusting; and simple shear.

In *Tensile Stress* the loads act outwards along the axis of the piece of material, giving rise to a tensile stress on any section

normal to the axis. The strain in this case consists of the amount the bar is stretched; this is shown by the dotted portion on Fig. 1 (*a*).

In *Compressive Stress* the line of action of the loads is the same, but they tend towards instead of away from one another,

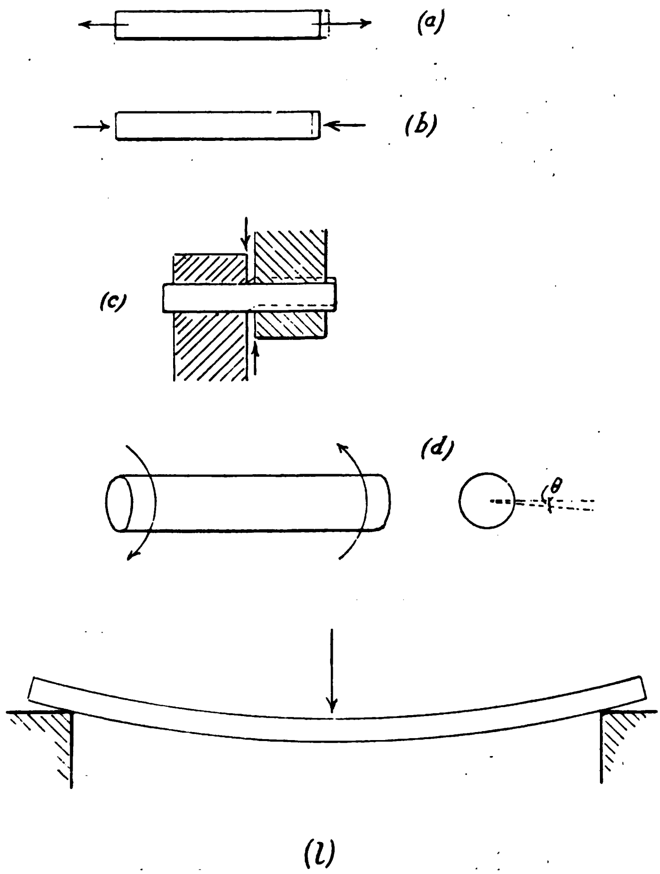

FIG. 1.

giving rise to a stress which compels the particles into closer union and at the same time causes a shortening, Fig. 1 (*b*).

The above are called *direct* stresses.

In *Simple Shear* the loads act parallel to one another in

opposite directions, tending to cause the two portions of the material acted upon to slide one upon the other.

The strain caused by shear stress is one of distortion. This is shown by the dotted portion, Fig. 1 (*c*).

Torsion is the particular case of shear stress which occurs when a shaft is twisted, Fig. 1 (*d*). The effect is to cause any two adjacent normal sections of the shaft to revolve relatively to one another. The strain in this case is measured by the angle of rotation of one end of the shaft with respect to the other.

Bending or Cross-breaking is the kind of stress met with where the load is applied in such a way that it causes a bar to bend, as shown on Fig. 1 (*l*). Here the bending action of the load results in a curving of the beam, the material on the convex surface being lengthened and put in tension, while that on the concave side is in compression. Thus there are two kinds of simple stress occurring at the same time. The strain in this case is the amount of deflection of the centre from its original position, measured in a direction at right angles to the axis.

Distribution of Stress—Uniform Stress.

It has been said that stress is expressed as the amount of the force (compression, tension, or shear) acting upon each unit of area exposed to it. When the load is caused to act in such a way that the intensity of the stress is the same at all points of the sectional area considered, it is said to be uniform. In such a case the stress is

$$f = \frac{P}{A}$$

where P is the total force or load and A the area of the section considered. This applies to simple tension, compression, and shear.

But it is possible to apply the load in such a way that the intensity of the stress is not the same at all points of the area. For instance, if the pull in a tie bar is applied along a line which lies outside the geometrical axis of the bar, the intensity of the stress on a section at right angles to the axis is not the same at all points. In such a case the above equation only serves to give the average stress.

Also, a stress may or may not vary uniformly. The principal cases of stress are summarised in the following table:—

Nature of Stress.	Kind of Strain.	How Strain is Measured.
Tension, Compression	Change in length	$\frac{\text{(Change in length)}}{\text{(Original length)}}$
Shear	Distortion	$\frac{\text{(Angle of distortion)}}{\text{(Unit angle)}}$
Torsion	Twisting	(Angle of twist)
Bending	Deflection	$\frac{\text{(Amount of deflection)}}{\text{(Span)}}$

Elasticity—Plasticity.—It has been said that the strain in a piece of material under stress is sometimes temporary and sometimes permanent, or partly temporary and partly permanent. If, after the removal of the load, the strain wholly disappears by reason of the material recovering its original form and dimensions, the strain is said to have been temporary and the state of the material to be *elastic.*

When, however, the material fails to recover its original dimensions and some of the strain remains after the removal of the stress, the material is said to have been strained beyond its elastic limit, and to have acquired *permanent set.*

Almost all the materials of engineering exhibit this property of taking permanent set after a certain stress has been reached. In some, permanent set is found after the application of very small stresses.

Materials like wrought iron and mild steel are found to arrive at a point in the loading when the strain goes on increasing with little or no increase in load. When this point has been reached the material is said to have become *plastic.*

In materials where this plastic state is reached, the period

extending from the end of the elastic to the beginning of the plastic stage is spoken of as the semi-plastic stage. The part of this subject which relates to the stresses and strains in the plastic and semi-plastic stages will be left for later consideration.

For the present, all the material dealt with will be assumed to be perfectly elastic, homogeneous and isotropic—that is, to have the same properties at all points and in every direction.

Hooke's Law.—The first law connecting stress and strain in elastic bodies is that enunciated by Hooke, and which says, "Stress is proportional to Strain." Thus, if a bar is strained a given amount under a certain stress, it will be strained double this amount under twice the stress, three times for thrice the stress, and so on. The same is true for every kind of strain, whether it be change in length, distortion, deflection or twist.

In Fig. 2 is shown a prismatic bar to which is applied a tensile load, P, acting along its axis. If the area of the cross-section of

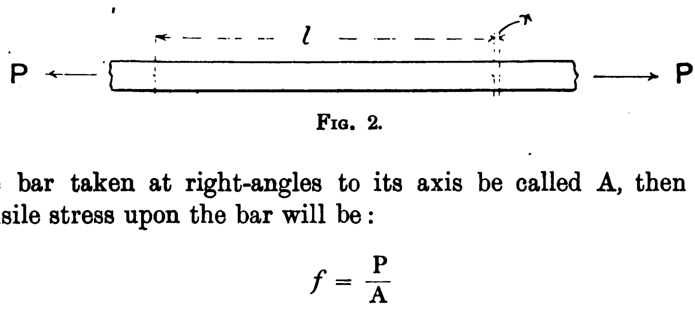

Fig. 2.

the bar taken at right-angles to its axis be called A, then the tensile stress upon the bar will be:

$$f = \frac{P}{A}$$

Previous to the application of the load two points are marked upon the surface of the bar, at a distance apart, l. When the load is applied this distance is increased by an amount x, so that the distance between the points is now $(l+x)$ instead of l.

The strain is then $\frac{x}{l}$, and as, according to Hooke's Law, stress is proportional to strain,

$$f \propto \frac{x}{l}$$

Young's Modulus for Direct Elasticity.—The amount of elastic strain in tension or compression corresponding to a given

stress varies for different materials, and is defined by what is called Young's Modulus, or the Elastic Modulus or Coefficient.

If the bar is extended an amount x, by the application of a stress f, then, according to Hooke's Law, in order to extend the bar an amount l, the stress must be proportionately greater. This stress is the elastic modulus and is constant for a given material.

It may be defined otherwise as the stress which would be required to extend (or compress) a body through a distance equal to its own original length, on the assumption that the material remained perfectly elastic.

This constant for direct elasticity is usually denoted by the symbol E. Thus:

$$\frac{x}{l} = \frac{f}{E}$$

or,
$$E = \frac{fl}{x}$$

or again,
$$E = \frac{\text{stress}}{\text{strain}} = \frac{f}{\frac{x}{l}}$$

where
$$\text{Strain} = \frac{\text{Variation in length}}{\text{Original length}} = \frac{x}{l}$$

If $\frac{x}{l}$, the strain, is made equal to unity, the stress f then becomes E, the elastic modulus, which from this point of view may be defined as the stress required to produce unit strain.

For example, if a bar of steel 1 sq. in. in section is found to stretch $\frac{1}{300}$th of an inch in 10 ins. of its length when a load of 10,000 lbs. is gradually applied, the value of the Young's Modulus will be

$$E = \frac{10{,}000 \times 10}{\frac{1}{300}}$$
$$= 30{,}000{,}000 \text{ lbs. per sq. in.}$$

This is somewhere near the usual modulus for steel and wrought iron, which is fairly constant, and is found to vary somewhere between 27,000,000 and 31,000,000 lbs. per sq. in. For cast iron it is about 15,000,000, and for copper 16,000,000 lbs. per sq. in.

In the above case the bar will have been stretched $\frac{1}{3000}$th

of its own length under a stress of 10,000 lbs. per sq. in. The working stress on structural steel does not often exceed 15,000 lbs. on the sq. in., or one-and-a-half times the above stress. This means that the strain will be increased in the same ratio, or the greatest stretch to be expected in steel under the working conditions will not exceed about $\frac{1}{2000}$th part of its length.

A knowledge of the modulus of any given material is important in enabling calculations to be made as to the strains likely to occur in structural parts when subjected to working loads, and also calculations of the probable strains in complex structures built of materials having widely differing moduli.

What has been said about elastic strain in tension applies equally to compression.

The uses and meanings of the terms which have been defined will be made clearer by following the details of the two examples given below. The figures given were found in making experiments upon bars of the materials mentioned.

Example 1.—Experiment on the elastic extension of a round wrought iron bar, 1 in. in diameter. The bar was placed in a testing machine, and loads applied in uniform increments. Extensions corresponding to the loads were measured by means of a Ewing's Extensometer, the units of whose scale correspond to $\frac{1}{500}$ths of an inch.

The following are the loads and the corresponding extensions as given by the extensometer readings:—

Loads—Tons	0	½	1	1½	2	2½	3	3½	4	4½	5
Readings .	0·00	0·11	0·31	0·51	0·71	0·92	1·12	1·33	1·53	1·74	1·95

Loads—Tons	5½	6	6½	7	7½	8	8½	9	9½	10
Readings .	2·15	2·35	2·56	2·76	2·97	3·18	3·40	3·62	3·83	4·05

These readings are shown plotted on the accompanying diagram (Fig. 3).

To find the elastic modulus, take pairs of readings having differences in loads of 5 tons:

4·05	3·83	3·62	3·40	3·18	2·97	2·76	2·56	2·35	2·15
1·95	1·74	1·53	1·33	1·12	·92	·71	·51	·31	·11
2·10	2·09	2·09	2·07	2·06	2·05	2·05	2·05	2·04	2·04

The total for ten readings being 20·64, the average extension for a difference in load of 5 tons is $\frac{2\cdot064}{500}$ ins.

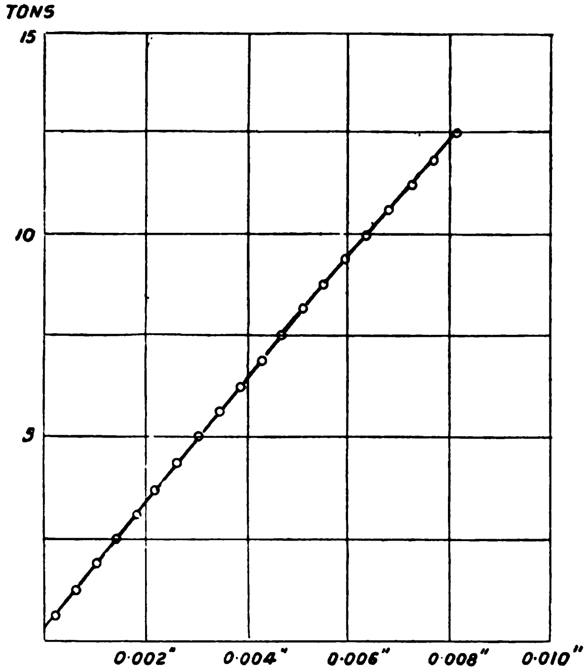

FIG. 3.—Load-extension diagram for a bar of wrought iron. Loads vertical. Extensions horizontal.

Since within the limits of the experiment strain is proportional to stress,

let f = stress in the material.
l = original length.
x = extension produced by f.
E = elastic modulus (Young's Modulus).

If the bar is extended an amount x for an increase of stress f, it will be stretched an amount l by a stress E, or,

$$E = \frac{fl}{x}, \text{ as before.}$$

$$f = \frac{5 \text{ tons} \times 2240}{0 \cdot 7854} \text{ lbs. per sq. in.}$$

l = 8 ins. (the measured length on the Ewing instrument).

$$x = \frac{2 \cdot 064}{500} \text{ ins.}$$

$$E = \frac{5 \times 2240 \times 8 \times 500}{0 \cdot 7854 \times 2 \cdot 064}$$

$$= 27{,}600{,}000 \text{ lbs. per sq. in.}$$

Example 2.—A compression experiment on a circular cast iron bar.

Diameter of bar 1·244 ins. Total length, 2·5 ins.

Readings were taken on 1·25 ins.

A Ewing's Extensometer was used to measure the compressions, the units of whose scale are $\frac{1}{2500}$ths of an inch.

Table of Loads and Compressions. (For diagram, see Fig. 4.)

Load increasing:

Load in tons	¼	2	4	6	8	10	12	14	16	18	20	22
Readings, $\frac{1}{2500}$ in.	0	0·6	1·4	2·22	3·03	3·84	4·7	5·6	6·57	7·73	9·18	11·3

Loads decreasing:

Loads in tons	22	20	18	16	14	12	10	8	6	4	2	¼
Readings, $\frac{1}{2500}$ in.	11·3	10·72	10·05	9·37	8·68	7·95	7·2	6·42	5·63	4·76	3·91	2·95

The bar was loaded up to 22 tons in steps of 2 tons, and the compressions noted (see first table). Then the loads were taken off and the extensions noted (see second table). To find the

average compression, the bar was taken as being elastic up to 12 tons, and the following differences were taken:

4·7	3·84	3·03
2·24	1·4	0·6
2·46	2·44	2·43

The average compression or difference in length for 6 tons load

$$= 2{\cdot}44 \text{ units,}$$

$$\text{which } = \frac{2{\cdot}44}{2500} \text{ ins.,}$$

$$= x$$

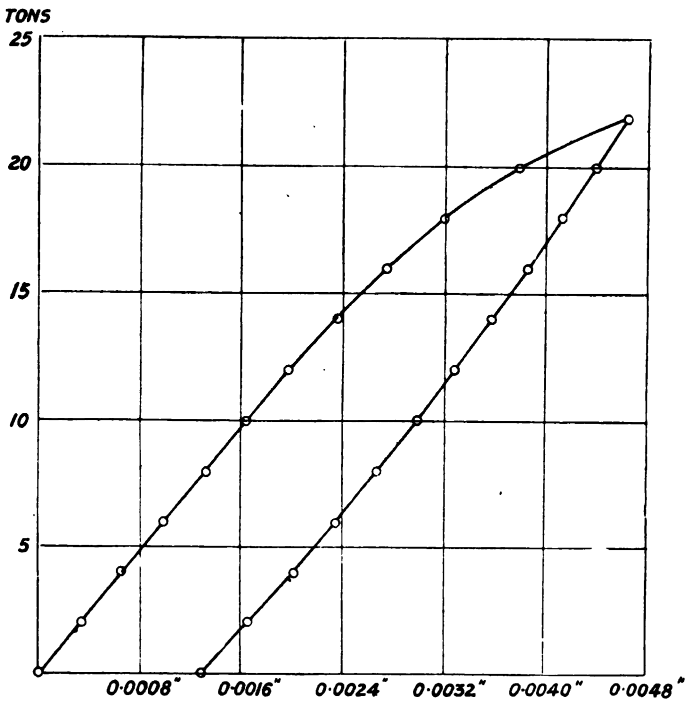

FIG. 4.—Load-compression diagram for a cast iron bar. Loads vertical. Compressions horizontal.

From which—

$$E = \frac{fl}{x}$$

$$= \frac{6 \times 2240 \times 1{\cdot}25 \times 2500}{(1{\cdot}244)^2 \times 0{\cdot}7854 \times 2{\cdot}44}$$

$$= 14{,}160{,}000 \text{ lbs. per sq. in.}$$

Example 3.—Find the amount of stretch of a copper trolley wire under a working stress of $\frac{1}{2}$ ton per sq. in. in a length of 400 yds.

Diameter of wire = $\frac{3}{8}$ in.
E = 16,000,000.

Working stress = $\frac{1}{2}$ ton per sq. in. = 1120 lbs. per sq. in.
Let x = amount of stretch in inches.

$$E = \frac{fl}{x}$$

$$\therefore \; x = \frac{fl}{E} = \frac{1120 \times 400 \times 3 \times 12}{16{,}000{,}000}$$

$$= 1{\cdot}008 \text{ ins.}$$

Answer: Amount of stretch = 1·008 ins.

The following two examples will serve to show how calculations are made which involve the use of E for two materials which are under stress at the same time:

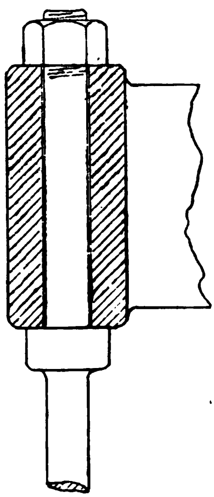

Fig. 5.

Example 4.—(See Fig. 5.)—A steel bolt 2 ins. in diameter passes through a circular casting 4 ins. in diameter and 10 ins. long. Find how far along the screw the nut will have to travel, after it just touches the casting, in order to put a tension of 5 tons per sq. in. on the bolt.

E for steel = 30,000,000 per sq. in.
„ cast iron = 15,000,000 „ „

Let the bolt be stretched an amount x_1
„ the casting be shortened an amount x_2
„ area of section of bolt be a_1 = 3·14 sq. ins.
„ area of section of casting be a_2 = 9·42 sq. ins.

The *total tensile load on the bolt* must equal the *total compressive load on the casting*. Let this

$$= W = f_t \times a_1 = f_c a_2$$

where f_t = the tensile stress on the steel,
f_c = the compressive stress on the cast iron.

$$x_1 = \frac{f_t l}{E_1} = \frac{5 \times 2240 \times 10}{30{,}000{,}000} = 0{\cdot}00373 \text{ in.}$$

and

$$x_2 = f_t \frac{a_1}{a_2} \frac{l}{E_2}, = \frac{5 \times 3{\cdot}14 \times 2240 \times 10}{9{\cdot}42 \times 15{,}000{,}000} = 0{\cdot}00248 \text{ in.}$$

The total movement of the nut down the screw is

$$x_1 + x_2 = 0{\cdot}00621 \text{ in.}$$

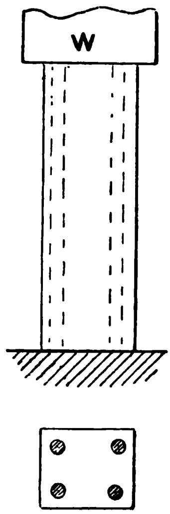

FIG. 6.

Example 5.—(See Fig. 6.)—A pillar of ferro-concrete is 3 ft. long, 10 ins. by 12 ins. cross-section. There are four bars of steel $1\frac{3}{4}$ ins. diameter running from end to end. The total load on the pillar is 50 tons. Find the load carried by the steel and the load carried by the concrete.

E for steel = 29,000,000 lbs. per sq. in.
E for concrete = 1,860,000 „ „

Let W_s, W_c be the loads on the steel and concrete respectively.
x_s, x_c be the compressions on the steel and concrete.

$$a_s = \text{Area of steel} = 0{\cdot}7854 \times (1{\cdot}75)^2 \times 4$$
$$= 9{\cdot}62 \text{ sq. ins.}$$
$$a_c = \text{Area of concrete} = 120 - 9{\cdot}62$$
$$= 110{\cdot}38 \text{ sq. ins.}$$

For the steel,

$$x_s = \frac{W_s \times 36}{9{\cdot}62 \times 29{,}000{,}000}$$

For the concrete,

$$x_c = \frac{W_c \times 36}{110{\cdot}38 \times 1{,}860{,}000}$$

but x_s must $= x_c$.

$$\therefore \quad \frac{W_s \times 36}{9{\cdot}62 \times 29{,}000{,}000} = \frac{W_c \times 36}{110{\cdot}38 \times 1{,}860{,}000}$$

or,
$$\frac{W_s}{W_c} = \frac{9 \cdot 62 \times 2900}{110 \cdot 38 \times 186} = \frac{1 \cdot 355}{1}$$

$$W_s = \frac{50 \times 1 \cdot 355}{2 \cdot 355} = 28 \cdot 77 \text{ tons.}$$

$$W_c = \frac{50}{2 \cdot 355} = 21 \cdot 23 \text{ tons.}$$

Lateral Contraction and Dilatation.—When an elastic body is extended under a tensile stress, it is found that a lateral shrinkage takes place. If the longitudinal strain is called θ, and the lateral strain—that is, the ratio of the diminution in thickness to the original thickness—is called ϕ, then it is found

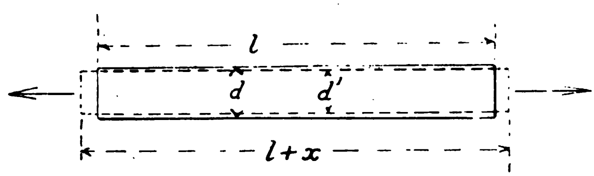

Fig. 7.—Poisson's Ratio.

that for a given material there is a constant relation existing between these two. Thus, if $\theta = \frac{x}{l}$ and $\phi = \frac{d-d'}{d}$, then $\mu = \frac{\theta}{\phi}$ where $\frac{1}{\mu}$ is known as "Poisson's Ratio." For indiarubber μ is found by experiment to have a value, for small strains, of 2. For hard solids it varies from 3 to 4, the latter value being that of the metals. It will be seen at a later stage that Poisson's Ratio becomes of especial importance when theoretical relations have to be established between Young's Modulus for direct stresses and the corresponding constants for shear and volumetric variation. The meaning of the above is illustrated on Fig. 7.

CHAPTER II

DIRECT, TANGENTIAL, AND OBLIQUE STRESSES

THE stress referred to so far has been either simple direct stress, as in tension and compression, or simple shear stress. If an imaginary plane surface in a body in a state of stress be taken anywhere in the material, the stress upon it may be either wholly normal, as in Fig. 8 (*a*); wholly tangential, as shown at (*b*); or it may be inclined to a normal to the surface at an angle α (*c*). The first two cases are quite simple. In the last case, the oblique stress f_r may be resolved into its two components, one acting

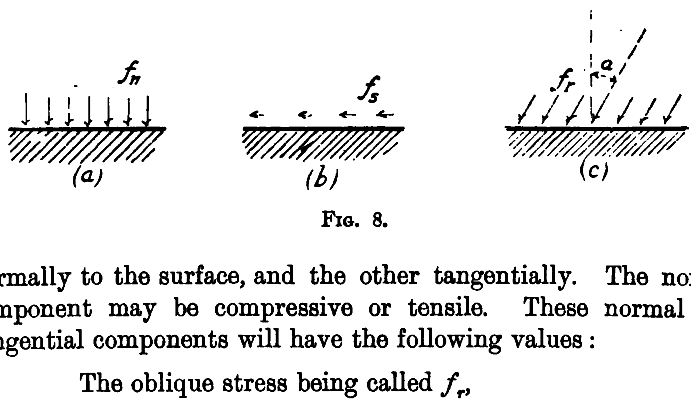

FIG. 8.

normally to the surface, and the other tangentially. The normal component may be compressive or tensile. These normal and tangential components will have the following values:

The oblique stress being called f_r,
The normal component stress $f_n = f_r \cos \alpha$, and
The tangential component stress $f_s = f_r \sin \alpha$.

Stress on an Area not at Right Angles to the Axis of a Body under Simple Tension or Compression.

In Fig. 9, a prismatic body is subjected to a direct load P along its axis, which gives rise to a simple direct stress on an

area XY normal to the axis. What will be the character and magnitude of the stress on any other area AB inclined to the first at an angle a?

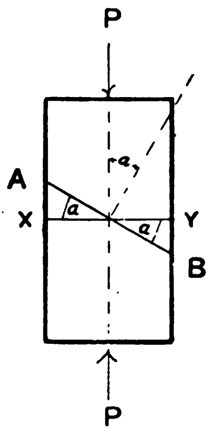

FIG. 9.

In the first place, the uniform direct stress on the normal surface XY is

$$f = \frac{P}{a}$$

where P is the whole direct load (compressive or tensile), and a is the area of the normal section XY.

The oblique stress on the surface AB will be

$$f_r = \frac{P}{\frac{a}{\cos a}} = \frac{P \cos a}{a}$$

This oblique stress can be resolved into its normal and tangential component stresses. Thus, the normal component

$$f_n = f_r \cos a = \frac{P \cos a \cos a}{a}$$
$$= f \cos^2 a$$

while the tangential component stress is

$$f_s = f_r \sin a = \frac{P \cos a \sin a}{a}$$
$$= f \sin a \cos a$$

This may be written, $\frac{f \sin 2a}{2}$. Sin $2a$ has a maximum value when $2a = 90°$, or $a = 45°$. So that the shear stress f_s has its greatest value when the angle between the two normals is 45°, and when

$$f_s = f \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} = \frac{f}{2}$$

Effect of two Direct Stresses at Right Angles.

Consider a cube, BCDE (Fig. 10), the area of whose face is (a). Let two pairs of normal forces, P′ and P″, act on pairs of opposite

faces, and let compressive forces be *plus* and tensile force *minus*. The intensity of stress on BC and ED is

$$f' = \frac{P'}{a}$$

and on BE and CD,

$$f'' = \frac{P''}{a}$$

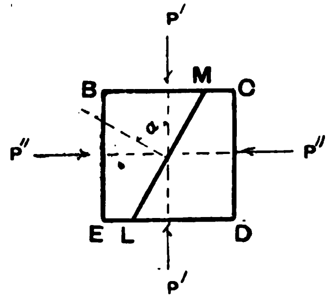

FIG. 10.

Consider the effect of these in producing stress on any other section LM taken through the centre and normal to the plane of the paper.

The normal stress on LM due to P′ is

$$f'_n = f' \cos^2 a$$

where a is the angle between the direction of P′ and the normal to LM. Similarly,

$$\begin{aligned} f''_n &= f'' \cos^2\left(\frac{\pi}{2} - a\right) \\ &= f'' \sin^2 a \end{aligned}$$

Therefore, the total normal stress on LM is

$$\begin{aligned} f_n &= f'_n + f''_n \\ &= f' \cos^2 a + f'' \sin^2 a \end{aligned}$$

Similarly, the tangential stress on LM due to P′

$$= f'_s = f' \sin a \cos a$$

tending to cause the part MBEL to slide in the direction M to L. Also the tangential stress due to P″ is

$$\begin{aligned} f''_s &= f'' \sin\left(\frac{3\pi}{2} + a\right) \cos\left(\frac{3\pi}{2} + a\right) \\ &= -f'' \sin a \cos a \end{aligned}$$

Therefore the total tangential stress on LM due to P′ and P″ is

$$\begin{aligned} f_s &= f'_s + f''_s \\ &= f' \sin a \cos a - f'' \sin a \cos a \\ &= (f' - f'') \sin a \cos a \end{aligned}$$

The value of this tangential stress attains a maximum when $\sin a \cos a$ is a maximum: that is to say, when $a = 45°$.

When the tangential stress is a maximum, and $a = 45°$, the total normal stress,

$$f_n = f' \cos^2 45° + f'' \sin^2 45°$$
$$= \tfrac{1}{2}(f' + f'')$$

The Nature of Shear Stress.—In Fig. 11 (a) is shown a piece of material rigidly held in sockets near its ends. These sockets are being forcibly moved, the upper one to the right and the lower one towards the left. The effect is to give rise to

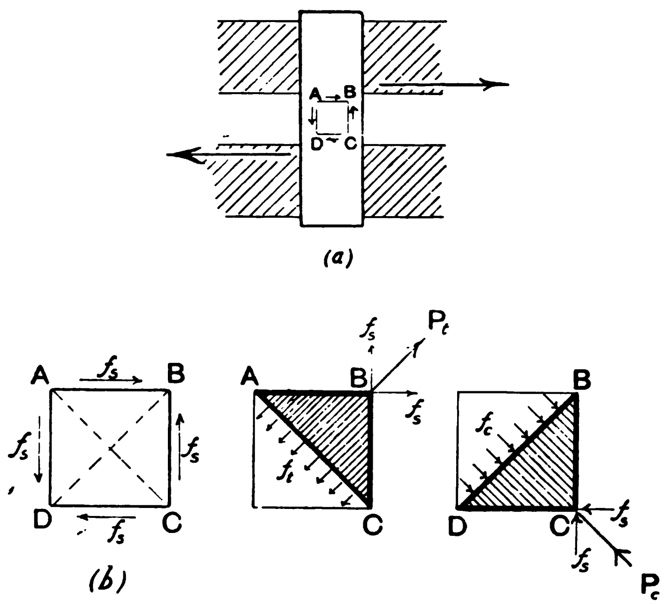

Fig. 11.

shearing stresses on planes at right angles to the axis of the bar, shown in the figure by AB and CD. Consider a small cube of the material between these two planes (Fig. 11 (b)). On the upper and lower faces of this cube there will be a pair of equal and opposite shearing stresses f_s, one acting in the direction AB and the other from C towards D. The tendency of this pair of stresses is to rotate the cube in the direction ABCD, there being a couple of a magnitude equal to the stress f_s, multiplied by the

vertical distance between AB and CD. This rotation does not take place, so that there must be a couple of equal magnitude acting in the opposite direction, to keep the cube in the given position. This couple consists of a second pair of stresses acting on the two faces, AD and CB, of the cube, and as the arm of this couple is equal to that of the first, it follows that the intensity of the stresses on AD and CB must be the same as those on AB and CD.

Next consider one pair of these four stresses acting on adjacent faces of the cube, namely, those on AB and CB respectively. These are forces of equal magnitude, acting in directions at right angles to one another. The direction of their resultant will bisect the angle between them, and, by the parallelogram of forces, will have a magnitude equal to

$$P_t = \sqrt{2}\, f_s \mathrm{AB}$$

Thus the pair of stresses will result in a single force, acting in a direction from D to B, and tending to pull half of the cube, ABC, away from the other half, ADC. In other words, this force, P_t, in tending to cause separation across the plane, AC, is exerting a tensile stress on this surface, which is resisted by the internal stress in the material, preventing separation of the surfaces from taking place.

The intensity of this tensile stress will be

$$\begin{aligned} f_t &= \frac{P_t}{(\text{area on which the stress acts})} \\ &= \frac{P_t}{\sqrt{2}\mathrm{AB}} = \frac{\sqrt{2} f_s \mathrm{AB}}{\sqrt{2}\mathrm{AB}} \\ &= f_s \end{aligned}$$

That is to say, there will be a tensile stress of equal intensity on a section of the cube taken through its diagonal, or being at an angle of 45° with the direction of the shearing stress.

By precisely similar reasoning it can be shown that the same shear stresses give rise to a compression stress, also of equal intensity to that of the shear stress, on a section taken through the other diagonal. This is shown on the figure.

The fact that these tangential and direct stresses do exist at the same time is very clearly demonstrated in the testing of

several kinds of material. For example, in the torsion test to destruction of a cast iron shaft, failure takes place, not by shearing, which is the original stress induced by the twisting, but by the material being torn asunder across a surface forming a spiral at an angle of 45° with the axis of the shaft. Here the tangential stress is accompanied by a tensile stress, and as the material is more ready to fail by pulling than by sliding, failure takes place in this way.

Another instance is to be found in the failure of most brittle materials under a crushing stress. The compressive stress induces shear stresses on planes inclined to the direction of the load, and the material usually fails by slipping along these planes. This is found to occur in crushing tests of such materials as cast iron, stone, cement, and so forth.

In mild steel tension tests the fractures most often take place on one plane forming an angle with the axis, or a pair of such planes forming a truncated pyramid and a corresponding recess. In the case of a round bar the fracture takes the form of an incomplete cone and cup.

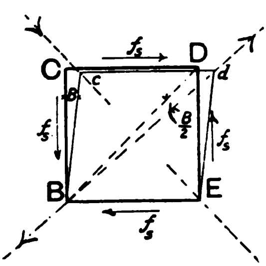

FIG. 12.

Shear Strain—Modulus of Rigidity.—Let a cube of unit length of side, one of whose faces is the square BCDE (Fig. 12), be subjected to a uniform shear stress f_s. It has been shown that on the pairs of opposite faces there will be stress f_s along CB, f_s along ED, f_s along EB, and f_s along CD. Also there will be a tensile stress f_t acting normally to the diagonal plane EC, and a compressive stress f_c on the diagonal plane BD. Moreover, it has been shown that

$$f_s = f_t = f_c$$

The effect of the shear stress is to distort the figure from the square to a new shape, B*cd*E, the lengths of the sides remaining unchanged. CB*c* is called the angle of distortion. Call this angle β. In elastic materials the angle of distortion is proportional to the shear stress, or

$$\beta \propto f_s$$

In order to produce distortion through an angle whose value in

circular measure is unity, it is necessary to apply a shear stress G where

$$\frac{G}{f_s} = \frac{1}{\beta}$$

$$\text{or} \qquad G = \frac{f_s}{\frac{\beta}{1}} = \frac{\text{shear stress}}{\text{angular strain}}$$

G is called the *Modulus of Rigidity*, or shear modulus.

It is clear that the angle

$$DBd = \frac{\beta}{2}$$

Next, to see how E and G are related.

It is obvious that when BCDE is distorted into the shape B*cd*E, the diagonal BD is lengthened so as to become B*d*. This lengthening may be considered to have been brought about partly by the tensile stress in the direction BD, and partly by the squeezing caused by the compressive stress in a direction at right angles to BD.

The elongation in direction BD due to f_t is

$$\frac{f_t BD}{E}$$

and in direction BD due to f_c is

$$\frac{f_c BD}{\mu E}$$

where $\frac{1}{\mu}$ is Poisson's ratio for the material in question. $BD = \sqrt{2}\,BC = \sqrt{2}\,.\,1$. So that the total elongation,

$$x = \frac{\sqrt{2}}{E}\left(\frac{f_t}{1} + \frac{f_c}{\mu}\right)$$

$$= \frac{\sqrt{2}}{E} f_s \left(\frac{\mu + 1}{\mu}\right)$$

Next, consider the lengthening of BD as caused by the distortion. This elongation is

$$x = Bd - BD$$

$$= 2BC \cos\left(\frac{\pi}{4} - \frac{\beta}{2}\right) - \sqrt{2}\,BC$$

$$= 2BC\left(\sin\frac{\pi}{4}\sin\frac{\beta}{2} + \cos\frac{\pi}{4}\cos\frac{\beta}{2}\right) - \sqrt{2}\,BC$$

But for very small angles,

$$\cos\frac{\beta}{2} = 1$$

$$\sin\frac{\beta}{2} = \frac{\beta}{2}$$

and here

$$BC = 1$$

so that

$$x = \frac{\beta}{\sqrt{2}}. \quad \text{But } \beta = \frac{f_s}{G}$$

and also

$$x = \frac{\sqrt{2}}{E} f_s \left(\frac{\mu + 1}{\mu}\right)$$

so that, finally,

$$\frac{f_s}{\sqrt{2}\,G} = \frac{\sqrt{2}}{E} f_s \left(\frac{\mu + 1}{\mu}\right)$$

$$\text{or} \quad \frac{G}{E} = \frac{\mu}{2(\mu + 1)}$$

In the case of the metals, where $\mu = 4$ (nearly),

$$\frac{G}{E} = \frac{2}{5}$$

That this is true may be shown by experiments in torsion and tension upon samples of the same material.

Elasticity of Volume or Cubic Elasticity (Fig. 13).—Let all six faces of the cube of unit length of side be pressed upon by a uniform stress p, which in this case is a pressure. The effect of this pressure will be to diminish the linear dimensions of the cube in all three directions, and in this way reduce its volume. If this decrease in volume be called v and the original volume V, then the volumetric strain will be $\frac{v}{V}$, and $\frac{v}{V} = \frac{p}{K}$, where K is a coefficient called the Modulus of Cubic Elasticity. This may be written

$$K = \frac{\text{pressure}}{\text{volumetric strain}} = \frac{p}{\frac{v}{V}}$$

$$\text{or} \quad K = \frac{pV}{v}$$

The following will show how K and E are related:—

Call the linear strain in any direction parallel to one edge of the cube x, due to the pair of pressures in the same direction. There will be a retardation or a dilatation in this direction, due to the other two pairs of pressures in directions at right angles to the first $= \frac{x}{\mu}$.

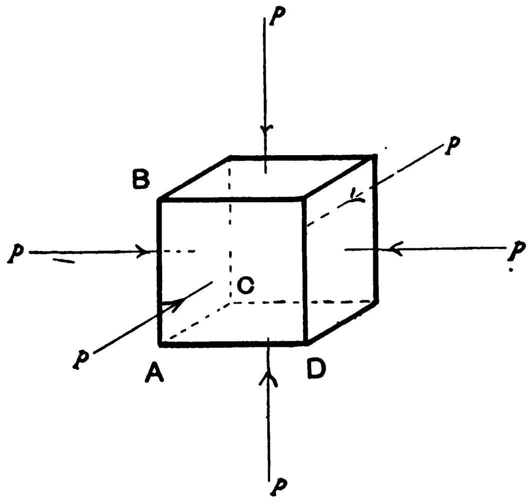

Fig. 13.

Thus the total strain in direction AB $= x - 2\frac{x}{\mu}$

" " " AC $= x - 2\frac{x}{\mu}$

" " " AD $= x - 2\frac{x}{\mu}$

For volumetric strain the total diminution is

$$3\left(x - \frac{2x}{\mu}\right)$$

and

$$K = \frac{pV}{3\left(x - 2\frac{x}{\mu}\right)}$$

but V = 1, so that here

$$K = \frac{p}{3\left(x - 2\frac{x}{\mu}\right)}$$

But $x = \frac{p}{E}$ for linear strain, so that

$$K = \frac{p}{3\left(\frac{p}{E} - \frac{2p}{E\mu}\right)}$$

$$= \frac{E\mu}{3\mu - 6}$$

If $\mu = 4$ $\qquad K = \frac{2}{3} E$

Relation between E, G, K.—It has previously been shown that

$$\frac{G}{E} = \frac{\mu}{2\mu + 2}$$

and it is now seen that

$$\frac{K}{E} = \frac{\mu}{3\mu - 6}$$

From these two equations the relation is obtained that

$$\mu = \frac{6K}{3K - E} = \frac{2G}{E - 2G}$$

or

$$E = \frac{9KG}{3K + G}$$

CHAPTER III

STRESSES IN BEAMS—BENDING AND SHEARING ACTIONS

THE effect produced by a force acting normally to the axis of a bar or prism is shown in Fig. 14.

Here is a bar of elastic material fixed or built into some rigid substance, and in this way forming a cantilever. The bar is supposed to have no weight. It is acted upon by a force, P, pushing upwards at its outer end and at right angles to its axis.

Consider any normal section of the cantilever at AB. The effect of P on the material in the plane AB will be twofold. In the first place, the tendency of P is to cause that portion of the cantilever to the right of AB to slide upwards, as shown at *b*, leaving the portion to the left stationary. The force, P, with which the part is pushed upwards is called the shearing force on AB. This causes shearing stress on the plane AB and this is opposed by the resistance offered by the material to sliding.

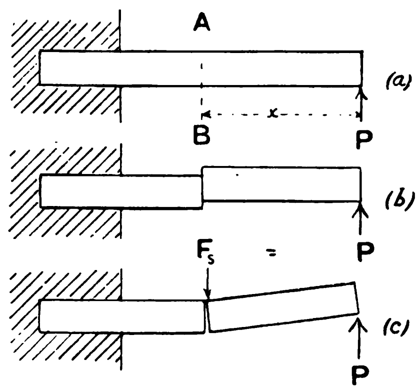

FIG. 14.

There will thus be shear stresses on the vertical section, and, from what has previously been said, it is clear that there must also be shearing stresses of equal magnitude on horizontal planes taken through AB.

The precise distribution of this shear stress over the section of the beam will be considered more in detail in a later paragraph.

For the present, it will be sufficient to investigate the second part of the effect of P.

If the beam were to be cut through at AB the shear effect of P would be to cause actual sliding to take place. When this movement is prevented by the application of a downward force $F_s = P$ acting at AB, it is evident that the further tendency of P will be to cause a separation between the planes at the lower part of the section, and, if there should be an opening between these at the top of the section, to close it. That is to say, there will be a tendency to sever the lower fibres by tension and to crush the upper fibres.

If x is the perpendicular distance from the line of action of P to the section AB, then the bending moment on the section is

$$M = P \times x$$

In the present simple case the bending moment is due to a

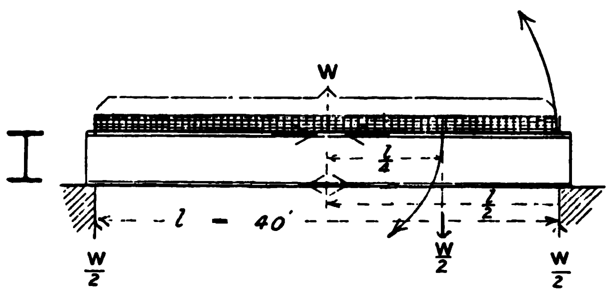

FIG. 15.

single force P. The bending moment on a section of a beam is in most cases due to a number of such forces, to a uniformly distributed load, or to both. It will be sufficient at present to consider the effect of a bending moment M, without any reference to the manner in which it is produced.

The following example will help to make clear what is the nature of the stress caused by bending in one of the simplest cases, namely, one in which the bending moment is resisted by stresses in the flanges of a girder.

Example.—A bridge, 40 ft. span and 12 ft. wide, is carried by two plate girders (Fig. 15). The total load per sq. ft. of platform is 7 cwts. Stress allowed in flange, $6\frac{1}{4}$ tons per

sq. in., depth of girder 4 ft. 6 ins., and width of flange 12 ins. Find the thickness of flange, t.

Total load on bridge $= \dfrac{(40 \times 12) \times 7}{20}$

$= 168$ tons.

Total load on one girder

$$= W = \frac{168}{2} = 84 \text{ tons.}$$

The bending moment is greatest at the centre, and is

$$= \frac{W}{2} \frac{l}{2} - \frac{W}{2} \frac{l}{4}$$

$$= \frac{Wl}{8}$$

This is equal to the moment of resistance

= (total force on flange) × (depth of girder)

= $(t \,.\, 12 \,.\, 6\frac{1}{4}) \times (54)$, inch-tons, where t is the required thickness.

Therefore

$$t\ 12 \,.\, 6\tfrac{1}{4} \,.\, 54 = \frac{Wl}{8}$$

or

$$t = \frac{84 \,.\, 40 \,.\, 12 \,.\, 4}{8 \,.\, 12 \,.\, 25 \,.\, 54}$$

$$= 1\tfrac{11}{45} \text{ ins.} = 1\tfrac{1}{4} \text{ ins. say.}$$

In the example just considered, it is assumed — which is approximately the case — that the tensile stress in the lower flange and the compressive stress in the top flange are both uniform over the flange section.

Where the section of the beam is solid, or where the thickness of the flange is great relatively to the depth of the beam, the fact that the stress varies from point to point must be taken into consideration.

Distribution of Direct Stress in a Beam Section.

When a beam is curved by bending, as seen in Fig. 16, it is evident that the layers of the material on the inside or concave surface are compressed and those on the convex surface are in a state of tension. Further, it is not difficult to see that these

stresses are greatest at the surfaces, and diminish as the centre of the section is approached.

If the portion to the right of the vertical section, CD, were to be hinged so as to be free to turn about a point, F, it would tend to revolve in the direction indicated by the arrow. This tendency is resisted by the compressive stresses exerted by the left-hand portion on that part of the section CF above F and by the tensile stress on FD, and the summation of the moments of these represents the moment of resistance, which is equal in magnitude but opposite in sense to the bending moment of the external forces.

Those stresses above F act to the right, and those below to

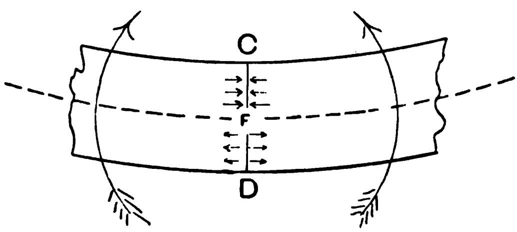

Fig. 16.

the left. As the external forces are wholly vertical, the algebraic sum of these horizontal stresses must be zero.

Now refer to Fig. 17.

At (*a*) two adjacent and parallel sections, AB and CD, are shown as they would appear on the unloaded beam. At (*b*) the same portion of the beam is shown when loaded.

The lines AB and CD, representing the sections, now appear to be no longer parallel, but inclined to one another and meeting in some point Q.

This convergence of the lines is brought about by the shortening of AC under the compressive stress, and the lengthening of BD under tension. C′D′ is drawn parallel to AB, at a distance away from AB equal to EF. CD and C′D′ intersect at F.

Consider the portion below EF. The fibres of the material at BD will have been extended an amount D′D, and in a fibre LM, at a distance x from EF, the extension is less. The material of the beam is supposed to be elastic, so that, as the stretch of the fibres diminishes in amount towards the centre, the stress also

diminishes, until some point F is reached where there is no longer any elongation and the stress is zero.

Beyond F the stress changes sign and becomes compressive, increasing in amount as the distance from F towards C increases.

The surface GH, which includes all points where there is no

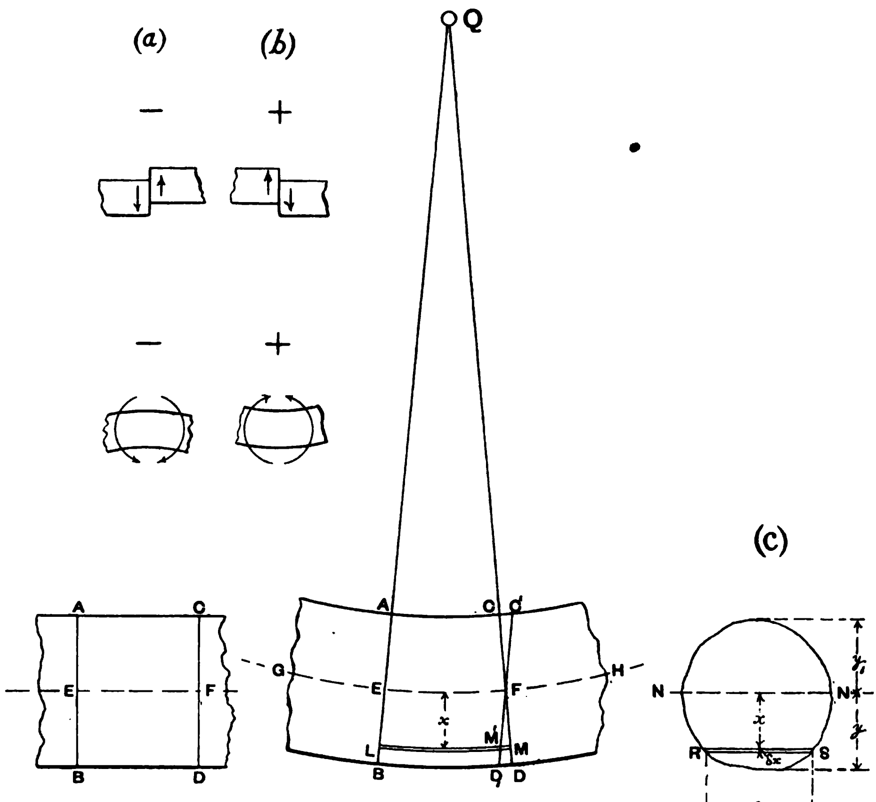

Fig. 17.

stress, is called the *Neutral Surface*, and its intersection with the plane section AB is a line NN, called the *Neutral Axis*.

In the theory relating to the bending stresses in beams the following assumptions are made:—

1. *The material remains elastic under all stresses imposed.*
2. *Its elastic modulus is the same for tension as for compression.*

For iron and steel experiment shows this to be practically true. In the case of such materials as concrete the difference is not great.

3. *A plane section of the beam, normal to the Neutral Surface, suffers no distortion when the beam is loaded but remains a plane.*

This means that the straight lines AB and CD in the figure remain straight when the beam is loaded. The extension (or compression) of any fibre LM, at a distance FM or x from the neutral axis F, is MM′, which forms the base of the triangle FMM′. This is one of a series of similar triangles having a common apex at F.

The lengths of their bases are proportional to their altitudes, x, and, as the stress is proportional to the extension (or compression), it follows that, if the above assumptions be sound, *the stress at any point must be proportional to the distance of that point from the neutral axis.*

In Fig. 17 (c) let RS be a strip of the section of the beam at AB, drawn parallel to the neutral axis NN, at a distance x from it, and let its length be a and its width δx.

If f is the maximum tensile stress at the edge of the section farthest from NN, and at a distance y from it, and q is the stress on the strip, then

$$\frac{q}{f} = \frac{x}{y}$$

$$\text{or} \qquad q = \frac{x}{y} f = cx$$

where c is a constant.

The total force on the strip is

$$q\,a\,\delta x = c\,x\,a\,\delta x$$

and the total force on the section is the integral

$$c\int a\,x\,dx$$

Below NN this is tension, and above NN it is compression; and as the whole of the stress on the section acts parallel to the neutral surface, and the external forces are wholly normal to this

direction, it follows that the algebraic sum of the direct stresses on the section must be zero, or

$$c\int a\,x\,dx = 0$$

From this it follows that the line from which x is measured, that is to say, *the neutral axis, must pass through the centre of gravity of the section*.

Moment of Resistance.—The force on one strip being

$$= q\,a\,\delta x$$

the moment of this force about the neutral axis is

$$= q\,a\,\delta x\,x = c\,a\,x^2\,\delta x$$

This is the *resisting moment* of one strip. The moment of resistance of the whole section is the integral

$$c\int a\,x^2\,dx$$

taken between the limits $x = -y$ and $x = +y_1$, y being the distance between the neutral axis and the point where the maximum stress f occurs, and y_1 the distance from the neutral axis to the extreme edge of the section opposite to y.

(Bending Moment) = (Moment of Resistance), or

$$M = c\int_{-y}^{+y_1} a\,x^2\,dx$$

but $c = \dfrac{f}{y}$, and the integral

$$\int_{-y}^{+y_1} a\,x^2\,dx$$

is the *Moment of Inertia* of the section about its neutral axis. Writing this as I, the above equation becomes

$$M = \frac{f}{y} I$$

$$\text{or} \quad \frac{M}{I} = \frac{f}{y}$$

The fraction $\dfrac{I}{y}$ is often called the *Modulus of the Section*, and is usually denoted by the symbol Z. The above equation then becomes

$$M = f\,Z$$

Moments of Inertia.—The moment of inertia of an element of area $\delta\omega$ about any axis is the product of the area $\delta\omega$ and the square of the perpendicular distance of its centroid from the axis. Thus in the case represented in the accompanying figure (Fig. 18) the moment of inertia

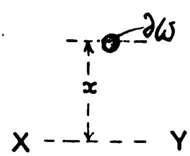

Fig. 18.

$$I = \delta\omega\, x^2$$

In the same way, the moment of inertia of an area which can be split up into a number of such elements is the summation of the moments of inertia of the several elements

$$I = \int x^2\, d\omega, \text{ or } = \int a\, x^2\, dx$$

where adx is the element of area.

Where the area in question is a regular geometrical figure the integration can generally be effected, and the value of I found mathematically. Thus in the case of the rectangle on Fig. 19, its I about its own neutral axis, or one drawn through its centre of gravity, is

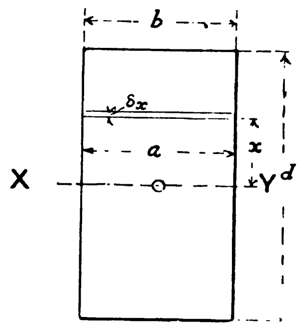

Fig. 19.

$$I = \int_{-y}^{+y} a\, x^2\, dx$$

when $a = b =$ constant, and $y = \dfrac{d}{2}$

Integrating,

$$I = b\left[\frac{x^3}{3}\right]_{-\frac{d}{2}}^{+\frac{d}{2}}$$

$$= \frac{bd^3}{12}$$

and the modulus

$$Z = \frac{I}{\frac{d}{2}} = \frac{bd^2}{6}$$

This case occurs very frequently and the result should be remembered.

The moment of inertia of any section may be written in the form,

$$I = A k^2$$

or

$$k^2 = \frac{I}{A}$$

where A = the area of the section in question,
and k = the *radius of gyration.*

The radius k is such that if the area A could be concentrated at the end of this radius, the effect would be the same.

The following rules are extremely useful when moments of inertia have to be found of areas whose figures are either rectangular or elliptical, or are complex areas made up of these figures.

1. The moment of inertia of an area about an axis through its centroid, and coinciding with or perpendicular to one of its principal axes, is

$$= \frac{[(\text{area of the figure}) \times (\text{the square of the rectangular semi-axis})]}{3 \text{ or } 4}$$

The divisor (3) is used where the figure is rectangular, and the (4) where it is elliptical.

Examples:—

(*a*) Moment of inertia of a circle about its diameter ($d = 2r$), Fig. 20.

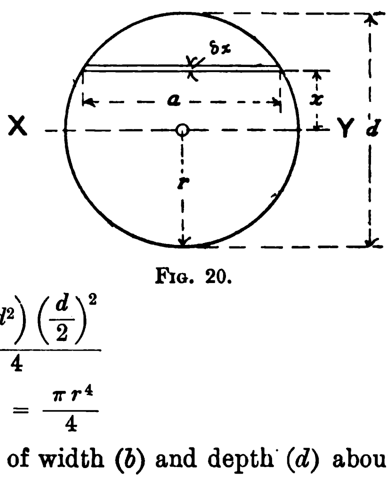

Fig. 20.

Here the semi-axes are the two radii, one in the neutral axis and one perpendicular to it. So that,

$$I = \frac{\left(\frac{\pi}{4} d^2\right)\left(\frac{d}{2}\right)^2}{4}$$

$$= \frac{\pi}{64} d^4 = \frac{\pi r^4}{4}$$

(*b*) Similarly, for a rectangle of width (b) and depth (d) about a neutral axis parallel to (b), Fig. 19.

$$I = \frac{(b\, d)\left(\frac{d}{2}\right)^2}{3}$$

$$= \frac{b\, d^3}{12}$$

2. The moment of inertia of an area about an axis other than its own, but parallel to it, is the sum of (the moment of inertia about its neutral axis) + (the product of its area into the square of the perpendicular distance between the two axes).

Thus (Fig. 21), if I is the moment of inertia of the given area about its neutral axis, A its area, h the perpendicular distance between the two axes, and I_1 the moment of inertia about the new axis XY, then

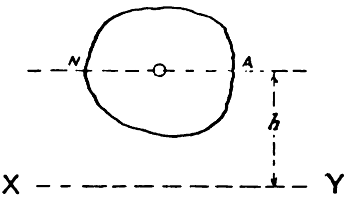

Fig. 21.

$$\begin{aligned} I_1 &= I + Ah^2 \\ &= Ak^2 + Ah^2 \\ &= A(k^2 + h^2) \end{aligned}$$

3. The moment of inertia of a complex area about any axis

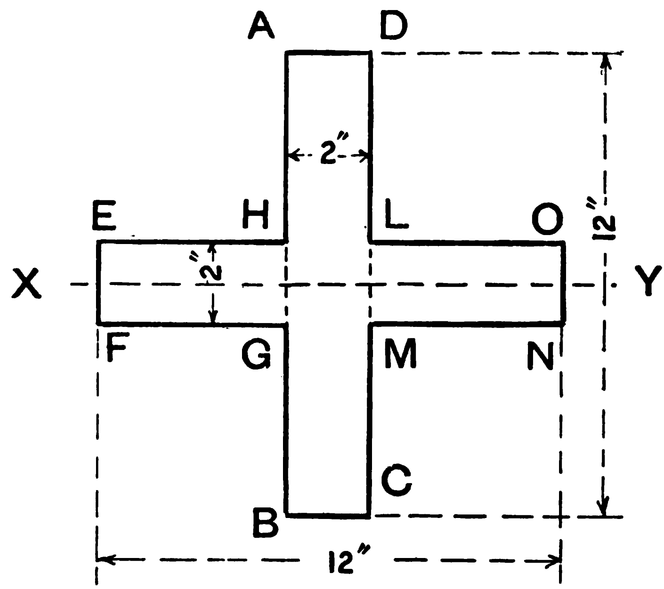

Fig. 22.

is the sum of the moments of inertia of the several parts about this same axis.

If I_1, I_2, I_3, etc., are the respective moments of inertia about the given axis of the several parts which make up the entire area, and I is the moment of inertia of the whole, then

$$I = I_1 + I_2 + I_3 + \text{ etc.}$$

The following worked-out examples should help the reader to a more complete appreciation of the rules which have just been laid down :—

Example 1.—To find the moment of inertia of the section (Fig. 22) about XY, its neutral axis.

Moment of inertia of rectangle ABCD about the axis XY . . . . $= \dfrac{2 \times (12)^3}{12} = 288$

Moment of inertia of rectangle LMNO about the axis XY . . . . $= \dfrac{5 \times (2)^3}{12} = \dfrac{10}{3}$

Moment of inertia of rectangle EFGH about the axis XY is same as that of rectangle LMNO . . . . . $= \dfrac{10}{3}$

Moment of inertia of entire figure about the axis XY . . . . $= \dfrac{10}{3} + \dfrac{10}{3} + 288$

$$= \frac{20}{3} + 288$$

$$= \frac{884}{3} = 294\dot{6} \text{ (inch-units)}$$

$$\text{Modulus of section} = \frac{\text{its moment of inertia}}{6}$$

$$= 49\dot{1}$$

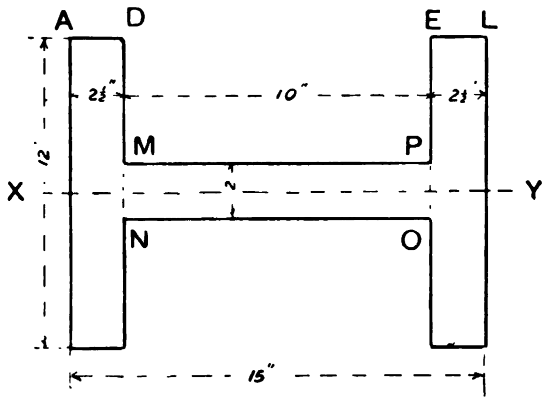

Fig. 23.

Example 2.—To find the moment of inertia of the given section about its neutral axis XY (Fig. 23).

Moment of inertia of rectangle AD about the axis XY . . $= \frac{2\frac{1}{2} \times (12)^3}{12} = \frac{5 \times 144}{2}$

Moment of inertia of rectangle EL about the axis XY . . $= \frac{2\frac{1}{2} \times (12)^3}{12} = \frac{5 \times 144}{2}$

Moment of inertia of rectangle MNOP about the axis XY . . $= \frac{10 \times (2)^3}{12} = \frac{20}{3}$

Moment of inertia of entire figure about the axis XY . . $= \frac{5 \times 144}{2} + \frac{5 \times 144}{2} + \frac{20}{3}$

$$= 720 + \frac{20}{3} = \frac{2180}{3} = 726 \cdot \dot{6} \text{ (inch-units)}$$

$$\text{Modulus of figure} = \frac{\text{its moment of inertia}}{6}$$

$$= \frac{726 \cdot \dot{6}}{6}$$

$$= 121 \cdot \dot{1}$$

Example 3.—To find moment of inertia of same section as last, but about neutral axis at right angles (Fig. 24).

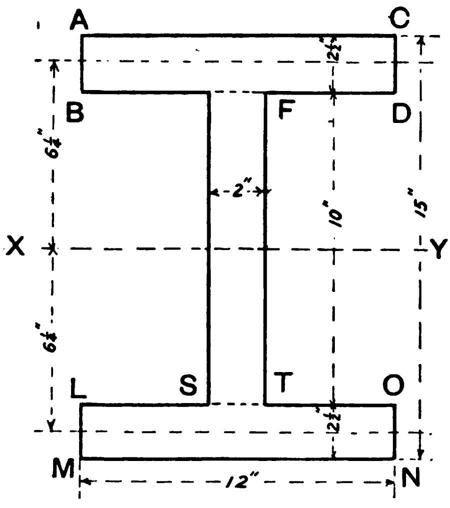

FIG. 24.

Moment of inertia of rectangle ABCD about the axis XY . $= \frac{12 \times (2\frac{1}{2})^3}{12} + (12 \times 2\frac{1}{2}) \times (6\frac{1}{4})^2$

$$= \frac{125}{8} + \frac{9375}{8} = \frac{9500}{8}$$

Moment of inertia of rectangle STF about the axis XY . $= \frac{2 \times (10)^3}{12} = \frac{1000}{6}$

Moment of inertia of rectangle LMNO about XY is the same as that of rectangle ABCD . $= \frac{9500}{8}$

Moment of inertia of entire figure about axis XY . . $= \frac{9500}{8} + \frac{9500}{8} + \frac{1000}{6}$

$$= \frac{9500}{4} + \frac{1000}{6}$$

$$= \frac{28500 + 2000}{12}$$

$$= \frac{30500}{12}$$

$$= 2581 \cdot \dot{6} \left(\begin{smallmatrix}\text{inch-}\\\text{units}\end{smallmatrix}\right)$$

$$\text{Modulus of section} = \frac{\text{moment of inertia}}{7\frac{1}{2}}$$

$$= 338 \cdot \dot{8}$$

Example 4 (Fig. 25).—Section of brickwork pier. To find its moment of inertia.

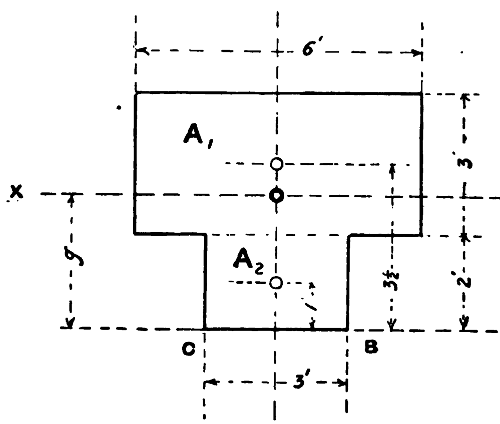

FIG. 25.

First, to find the centre of gravity or centroid in order to fix the position of the neutral axis.

Taking moments about BC,

$$(\text{Entire area}) \times g = (\text{Area } A_1) \times 3\tfrac{1}{2} + (\text{Area } A_2) \times 1$$

or $$g = \frac{18 \times 3\frac{1}{2} + 6 \times 1}{24} = \frac{69}{24}$$

$$= 2{\cdot}875 \text{ feet}$$

The moment of inertia of the entire area about XY is

$$\begin{aligned}
\mathrm{I} &= (\text{Moment of I of } \mathrm{A}_1) + (\text{Moment of I of } \mathrm{A}_2) \\
&= \left[\frac{b\,d^3}{12} + b\,d(0{\cdot}625)^2\right] + \left[\frac{bd^3}{12} + b\,d\,(1{\cdot}875)^2\right] \\
&= \left(\frac{6 \times 27}{12} + 18 \times 0{\cdot}39\right) + \left(\frac{3 \times 8}{12} + 6 \times 3{\cdot}52\right) \\
&= (13{\cdot}50 + 7{\cdot}02) + (2 + 21{\cdot}1) \\
&= 43{\cdot}60 \text{ feet-units.}
\end{aligned}$$

CHAPTER IV

GRAPHICAL METHOD FOR DETERMINING THE MOMENT OF INERTIA

WHERE the shape of the area whose moment of inertia is required is that of one of the simple geometrical figures, such as a square, rectangle, triangle, circle, or the combination of a number of these, there is little difficulty in determining the required value by the methods which have just been described. But in many cases which occur in practice the area is of such irregular shape that the problem cannot be attacked mathematically, and a graphical method, depending upon the following principles, must be employed.

It has previously been shown (see Fig. 17) that the moment of resistance is equal to the integral

$$\int \frac{f}{y} a\, x^2\, dx$$

Here, it will be remembered, f is the maximum stress at a distance y from the neutral axis, and a is the length of the strip which forms the element of area. The above integral may be written

$$\int \left[\frac{x}{y} f\right] [a\, x\, dx]$$

The actual stress on the element in question is $\frac{x}{y} f$, or f reduced in the proper ratio, $\frac{x}{y}$.

Now, without altering its value, the same integral may be written as

$$\int \left[\frac{x}{y} a\right] [f\, x\, dx]$$

The meaning suggested by this way of expressing it, is that instead of reducing the *stress* in the ratio of $\frac{x}{y}$, the length of the *strip* is reduced in the same ratio, and the stress is taken as having the constant value f. In other words, the force due to a *reduced stress* acting on the *actual element of area* ($a\ \delta x$) is replaced by a force consisting of the *original maximum stress* acting on a *reduced area.* The ratio of reduction is the same in both cases, so that the total force on the strip must be the same.

If this reduction is made in each element of area of which the original section is composed, these new elements added together will make up a new figure, called the modulus figure. The

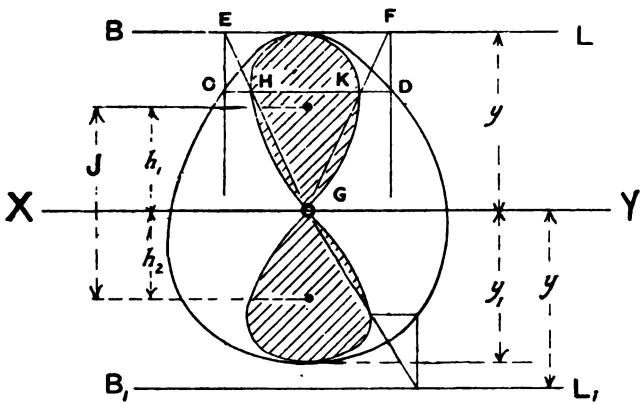

Fig. 26.

original maximum stress f acting on this area will give a moment of resistance equal to the one represented by the above integral.

The resultant of the uniform stress on the modulus figure must pass through the C.G. of this figure.

The moment of resistance on one side of the neutral axis will then be

$= f \times$ (area of modulus figure) $\times$ (distance of its C.G. from the neutral axis)

$= f \times Z$

The actual way of finding the modulus graphically is as follows:—

Consider any figure as shown in the accompanying sketch, Fig. 26.

(1) Find the centre of gravity. This is most conveniently done by cutting out the figure in thin cardboard and balancing it above a knife edge in two positions, roughly at right angles to one another. The intersection of the two lines about which the figure balances will be the required C.G.

(2) Draw the neutral axis XY through the C.G. at right angles to the direction in which the load is applied.

(3) Draw a *base-line* BL parallel to the neutral axis and touching the figure at the point distant farthest from XY, this distance being y. Also draw a second base-line BL on the other side of XY and the same distance, y, from it.

(4) Draw a number of lines, as CD, parallel to XY and cutting the boundaries of the figure at C and D. Drop perpendiculars CE and DF on to BL.

Join E and F to G. The lines thus drawn will cut CD in H and K.

C and D are points on the boundary line of the original figure; H and K will be the corresponding points on the modulus figure. The remaining points are to be found in the same way. By following out this construction throughout, the two shaded modulus figures will be obtained. These are such that a "constant stress upon them equal to the maximum stress f will have the same effect as the varying stress on the original figure," by reason of the proof already given.

(5) As the nature of the stress on one of these areas is tensile and on the other compressive, and as the algebraic sum of the forces acting at any point of the beam parallel to its axis is zero, it follows that the area of the modulus figure above XY must be equal to the portion below XY.

(6) Measure the two areas. This is most conveniently done by using a planimeter.

Call the area above XY ω_1, and the one below XY ω_2. When the two areas have been measured, ω_1 should be equal to ω_2. The difference ought not to be more than 1 per cent.

This equality of the two areas provides an excellent check on the accuracy of the previous work. When their difference is found to be too great, the finding of the C.G. of the

original figure and the construction used for obtaining the modulus figures must be revised or repeated.

(7) Find the centres of gravity of the two modulus figures. It will be found most convenient to cut out the figures in card-board and balance, as in the case of the original figure. Call the distances of these C.G.'s from XY respectively h_1 and h_2.

Then the moment of resistance about XY will be

$$= f\,\omega_1 h_1 + f\,\omega_2 h_2$$

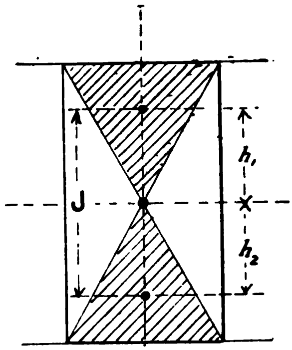

FIG. 27.

the resultant of the total force on each modulus figure acting at its C.G., and the arms of the moments being h_1 and h_2.

But $\omega_1 = \omega_2 = \omega$, so that the moment of resistance is

$$= f\,\omega\,(h_1 + h_2)$$
$$= f\,\omega\,J$$

Here the modulus Z, is represented by $\omega \times J$.

It is rarely that ω_1 comes out quite equal to ω_2. When they are sufficiently alike, ω must be taken as their mean value.

To find the moment of inertia when the modulus has been determined, it is only necessary to multiply Z by the distance from the neutral axis to the base-line, or

$$I = Zy$$

Where the figure is symmetrical about the neutral axis, $y = y_1$ and the base-lines touch the figure at both the top and the bottom.

Several examples of the modulus figures for symmetrical areas are given below. Of these,

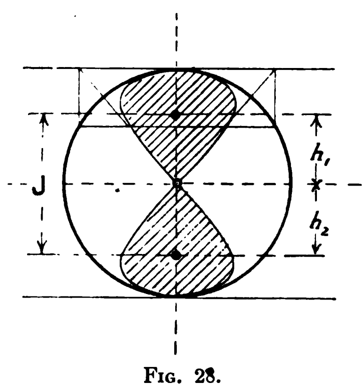

FIG. 28.

Fig. 27 shows the modulus figure for the rectangle shown on Fig. 19.

Fig. 28 shows the modulus figure for the circle shown on Fig. 20.

Fig. 29 shows the modulus figure for the section shown on Fig. 22.

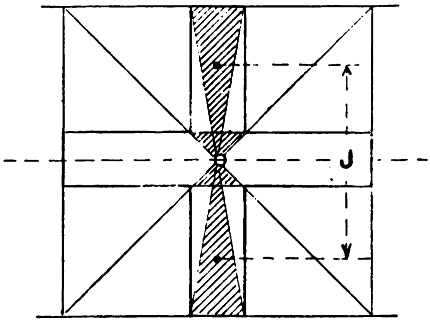

Fig. 29.

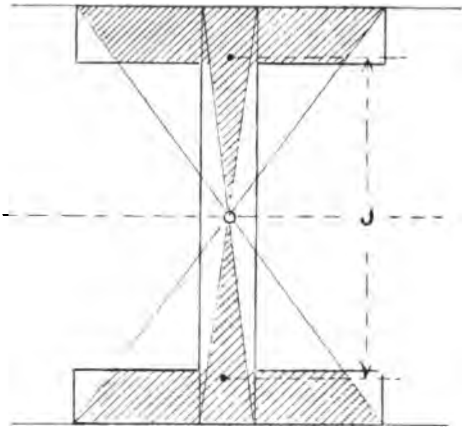

Fig. 30.

Fig. 30 shows the modulus figure for the section shown on Fig. 24.

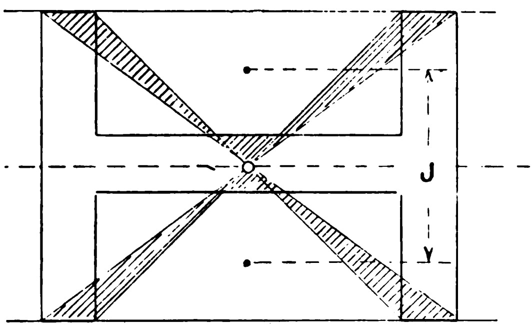

Fig. 31.

Fig. 31 shows the modulus figure for the section shown on Fig. 23.

In cases where the original figure is not symmetrical about the neutral axis, as in Fig. 26, the modulus figures must be constructed from base-lines at a distance from the neutral axis corresponding to the point where the maximum stress of greatest importance occurs.

For instance, the area on Fig. 32 represents the section of a

cast iron frame. Cast iron is known to be much weaker in tension than in compression, and it is therefore necessary to design the section on the basis of a maximum safe tensile stress. This occurs at the edge of the large flange, and the base-line must be taken at the distance of this edge from the neutral axis.

The modulus will have a value differing according to which base-line is used.

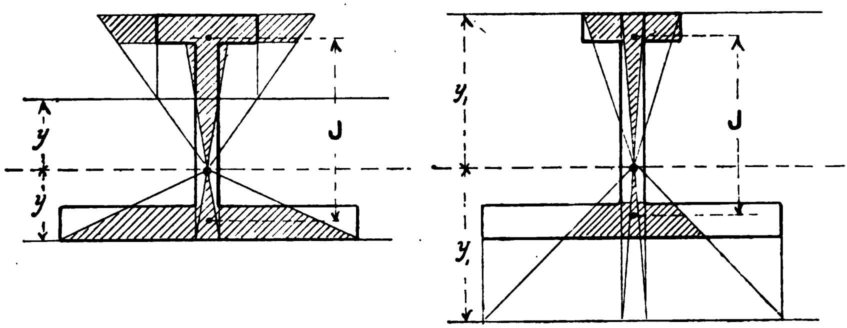

FIG. 32.

It is to be noted, however, that the value of the moment of inertia will be the same whichever base-line is used.

Thus, in an unsymmetrical section, let

y = distance of one base-line from neutral axis;
Z = the modulus as found from this;
y_1 = distance of other base-lines from neutral axis; and
Z_1 = the corresponding modulus.

Then, the moment of inertia

$$I = Zy = Z_1 y_1$$

From which

$$\frac{Z_1}{Z} = \frac{y}{y_1}$$

or

$$Z_1 = Z\frac{y}{y_1}$$

In this way Z_1 can be found from Z by multiplying it by the inverse ratio of the y's.

The two pairs of modulus figures giving Z and Z_1 are shown for the section in Fig. 32.

The examples given here of the finding of the modulus in the cases of areas made up of regular geometrical figures are chiefly given for the purpose of showing how the method is carried out.

As a rule, for figures of this type it is much easier, and, at the same time, more accurate, to find the modulus by calculation in the manner already explained.

On Fig. 33 is shown a good typical example of the kind of

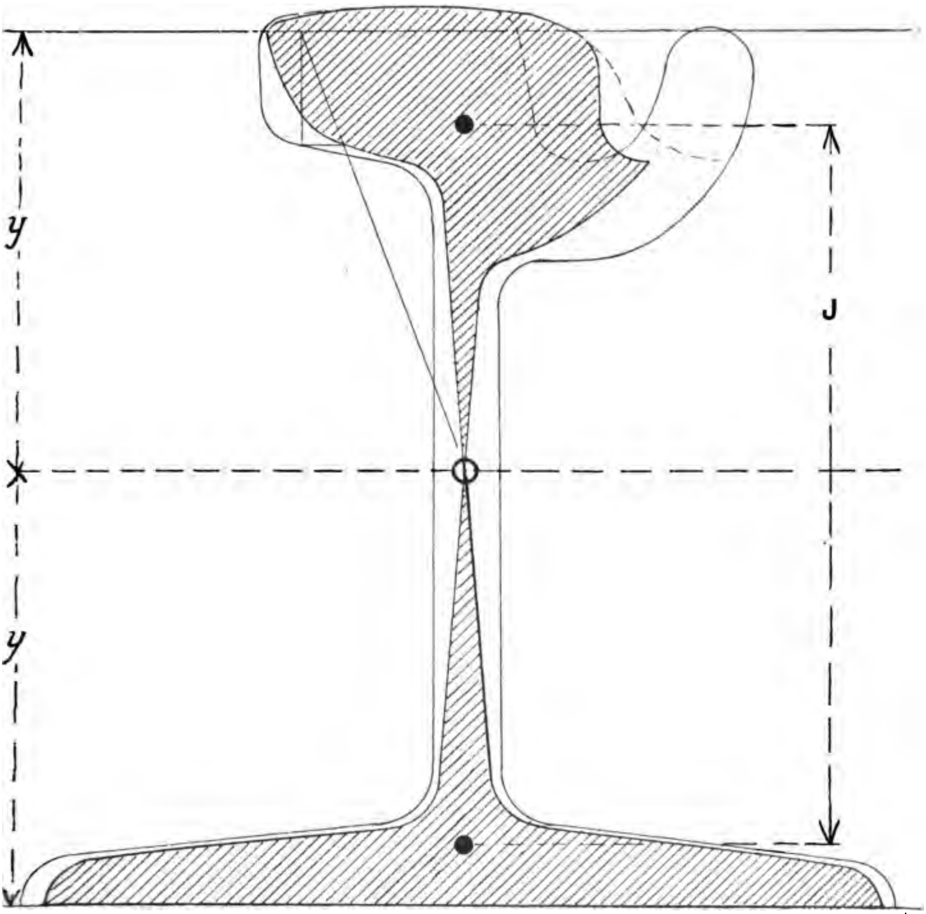

FIG. 33.

area in which it is not possible to find Z by calculation, namely, that of a tram-rail section.

The modulus has in this case been found as described, working from the tension base-line at the lower edge of the flange.

It is sometimes found convenient, though not necessary, to

"mass together" a separated section. In the above case it is split by the groove in which the flange of the wheel runs.

This "massing" is done by drawing a number of horizontal lines across the part to be dealt with, and transferring the outstanding widths to the solid part, as indicated in the drawing. The new part thus formed is shown by the dotted line. This process does not alter the value of the modulus about a neutral axis parallel to these horizontal lines.

The actual measurements taken from Fig. 33 were as follow :—

Area of modulus figure.	*Top.*	*Bottom.*	
Three measurements were made on each side the neutral axis	3·89	3·89	
	3·87	3·90	
	3·89	3 91	
	3·88	3·90	means

The average of the two means so obtained is 3·89 sq. ins. J was measured as 5·61 ins., and y as 3·425 ins. From these the modulus

$$\begin{aligned} Z &= 3{\cdot}89 \times 5{\cdot}61 \\ &= 21{\cdot}82 \text{ inch-units,} \end{aligned}$$

and the moment of inertia,

$$\begin{aligned} I &= Z \times y \\ &= 21{\cdot}82 \times 3{\cdot}42 \\ &= 74{\cdot}74 \end{aligned}$$

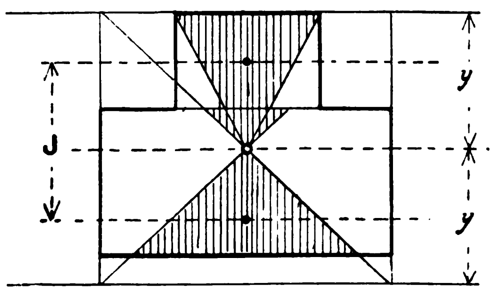

FIG. 34.

The modulus figure shown on Fig. 34 is for the same section as that previously given on Fig. 25.

CHAPTER V

DEFLECTION OF BEAMS

IN this connection the first thing to be done is to find an expression for the radius of curvature R at any point of the deflected beam, this curvature generally varying from point to point.

In Fig. 35, which is similar to Fig. 17, the two plane sections AB and CD, originally parallel, are taken sufficiently near to one another to make the radius of curvature sensibly constant from E to F. The effect of the bending is to cause AB and CD to become inclined to one another and to meet in Q, which is the centre of curvature for the short length considered. The bottom fibres are lengthened an amount D′D.

Call BD′ = EF = AC′ = L, and D′D = s.

Then $\frac{s}{L} = \frac{f}{E}$, according to Hooke's Law.

Now C′D′ has been drawn parallel to AB, so that the triangles QEF and FD′D are similar.

It therefore follows that $\frac{EF}{QE} = \frac{D'D}{FD'}$

Here,

$$\begin{aligned} EF &= L, \\ QE &= R \text{ (at the neutral surface)}, \\ DD'. &= s, \text{ and} \\ FD'. &= y, \end{aligned}$$

so that the above may be written,

$$\begin{aligned} \frac{1}{R} &= \frac{s}{Ly} \\ &= \frac{1}{E}\frac{f}{y}\left[\text{but } \frac{f}{y} = \frac{M}{I}\right] \\ &= \frac{M}{EI} \end{aligned}$$

This equation is fundamental in relation to the deflection of beams. Its meaning is that the reciprocal of the radius of curvature of the neutral surface at any point in a loaded beam is equal to the bending moment at this point divided by the product of the elastic modulus of the material and the moment of inertia of the section.

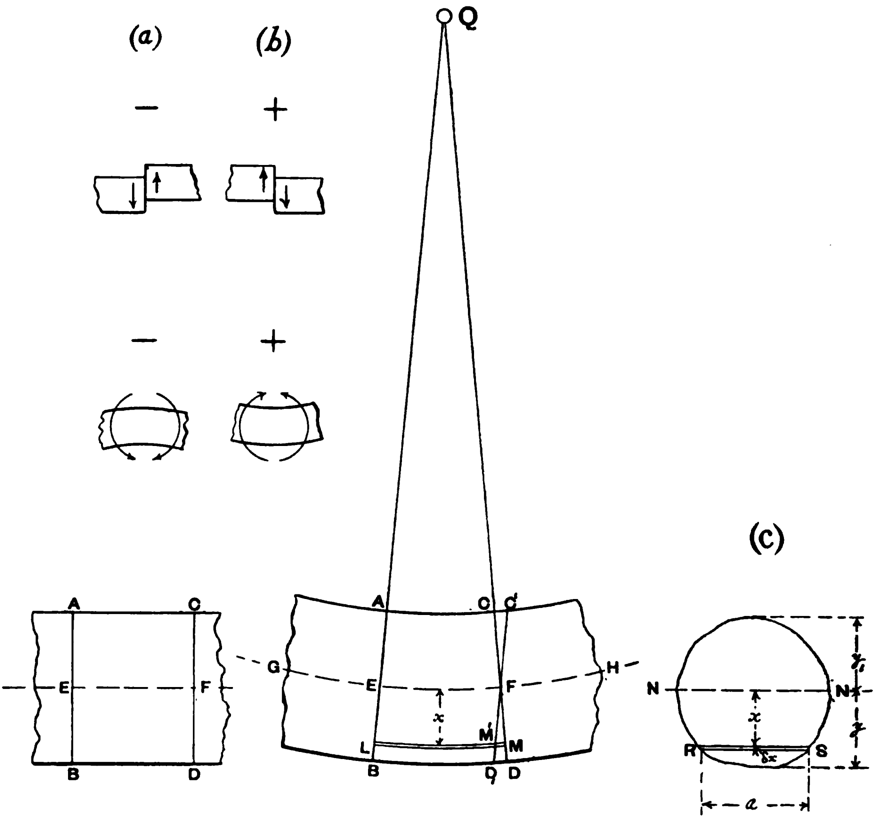

Fig. 35.

It is necessary here to take note of the following definitions:—

1. *The shearing force* S *at any section of a beam is the algebraic sum of all the forces normal to its axis acting on one side of the section.*

   For the sake of uniformity, those shearing forces which act as at (*a*) (Fig. 35) are taken as *minus*, and those as at (*b*) as *plus*.

2. *The bending moment* M *at any section of a beam is the algebraic sum of all the moments acting on the beam on one side of that section.*

Those as at (a) are *minus*, and as at (b) are *plus*.

Next, to find the deflection. There are two ways of doing this.

First Method.—CASE I. *Deflection at the end of a cantilever carrying one concentrated load.*

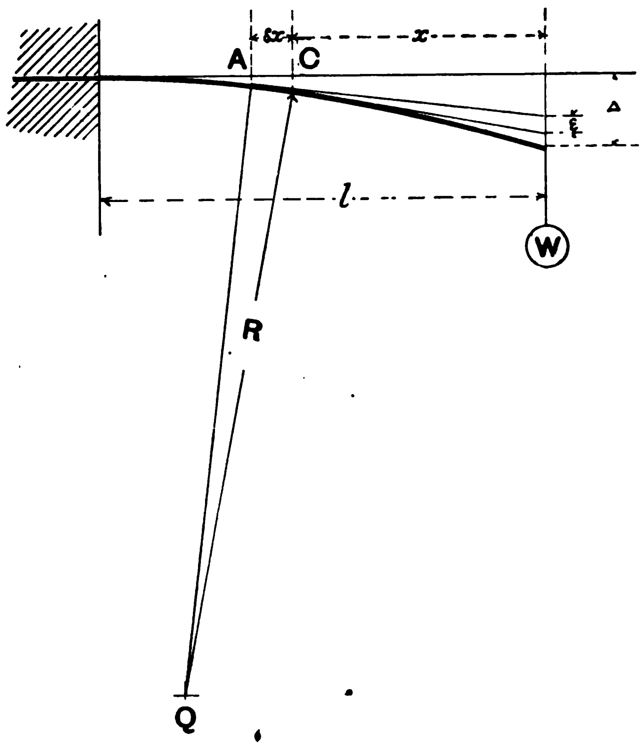

FIG. 36.

It is only necessary in what follows to consider the curvature of the neutral surface (shown by strong lines in the figures).

On Fig. 36 is shown the cantilever in question, having a length l and carrying a single load W at its outer end. Take any two adjacent points A and C at a horizontal distance x from the outer end, and at a distance apart δx.

Radii are drawn from A and C to meet in Q, and corresponding tangents from these points cut off from the vertical line through the outer end of the cantilever a short length, ϵ. This will be the deflection at the outer end due to the curvature of the element of length δx. The total deflection $\triangle$ will be the summation of all these increments of deflection, from a point where $x=0$ to $x=l$.

Now, by similar triangles,

$$\frac{\delta x}{\mathrm{R}} = \frac{\epsilon}{x}$$

or $$\frac{1}{\mathrm{R}} = \frac{\epsilon}{x\,\delta x}.$$

but it has already been shown that

$$\frac{1}{\mathrm{R}} = \frac{\mathrm{M}}{\mathrm{E\,I}}$$

Therefore $$\epsilon = \frac{\mathrm{M}\,x\,\delta x}{\mathrm{E\,I}}$$

At the point in question, the bending moment,

$$\mathrm{M} = -\mathrm{W}\,x$$

so that

$$\epsilon = -\frac{\mathrm{W}\,x^2\delta x}{\mathrm{E\,I}}$$

The total deflection is the integral of this, or,

$$\triangle = -\frac{\mathrm{W}}{\mathrm{E\,I}}\int_0^l x^2\,dx$$

$$= -\frac{\mathrm{W}l^3}{3\,\mathrm{E\,I}}$$ The *minus* sign indicates a downward deflection.

CASE II. *Cantilever loaded uniformly.* (Fig. 37.)

In this the load is applied uniformly and is equal to w units of weight per unit of length. As before,

$$\epsilon = \frac{\mathrm{M}\,x\,\delta x}{\mathrm{E\,I}}$$

But the bending moment at the point AC is

M = (load on portion, x) × (distance of the C.G. of this load from the section),

or $$= -(w\,x)\left(\frac{x}{2}\right) = -\frac{w\,x^2}{2}$$

so that $$\epsilon = -\frac{w\,x^3\,\delta x}{2\,E\,I}$$

and the whole deflection is

$$\begin{aligned}\triangle &= -\frac{w}{2\,E\,I}\int_0^l x^3\,dx \\ &= -\frac{w\,l^4}{8\,E\,I} \quad \text{but } w\,l = W \text{ where W is the total load.} \\ &= -\frac{W\,l^3}{8\,E\,I}\end{aligned}$$

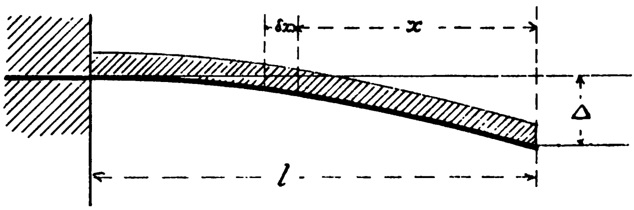

FIG. 37.

The same method can be used for supported beams, or the above two results may be utilised as follows:—

CASE III. *Beam freely resting upon two supports whose distance apart is* l, *and carrying a concentrated load* W *at the centre* (Fig. 38).

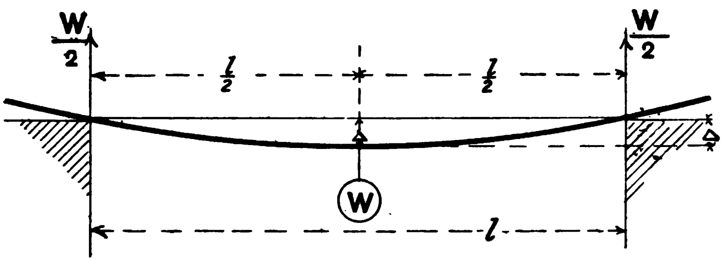

FIG. 38.

The upward reaction at each support is half the load, or, $\frac{W}{2}$, and the neutral axis is horizontal at the centre The beam can therefore be regarded as made up of two similar inverted cantilevers, of length $\frac{l}{2}$, and loaded at their outer ends with up-

ward forces $=\frac{W}{2}$. The vertical distance between the centre and the points which touch the supports will, from the former of the above equations, be

$$\triangle = \frac{\left(\frac{W}{2}\right)\left(\frac{l}{2}\right)^3}{3\,E\,I}$$

$$= \frac{W\,l^3}{48\,E\,I}$$

CASE IV. *Beam resting upon two supports whose distance apart is l, and carrying a uniformly distributed load w per foot run* (Fig. 39).

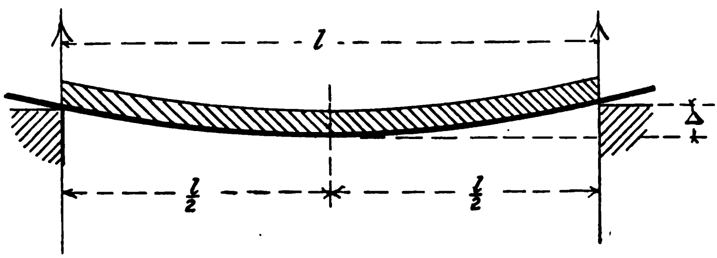

FIG. 39.

The total load $W = w\,.\,l$

The beam is symmetrical about the centre, where the maximum deflection will occur. The reactions at the support $=\frac{W}{2}=\frac{wl}{2}$. The beam can be taken as a pair of cantilevers, each starting from the centre and having an upward force $\frac{W}{2}$ at the outer end or support, deflecting it upwards, and a uniform load $\frac{W}{2}$ tending to deflect it downwards.

The required deflection $\triangle$ will be the difference between the upward and the downward deflections of the cantilever, whose length is $\frac{l}{2}$, that is,

$$\triangle = \frac{\left(\frac{W}{2}\right)\left(\frac{l}{2}\right)^3}{3\,E\,I} - \frac{\left(\frac{W}{2}\right)\left(\frac{l}{2}\right)^3}{8\,E\,I}$$

$$= \frac{5}{384}\;\frac{W\,l^3}{E\,I}$$

CASE V. *Beam fixed horizontally at the supports and carrying a single concentrated load* W *at the centre. As before, the span is l.* (Fig. 40.)

It is necessary that the neutral axis be held perfectly horizontal at the supports, and that these be on the same level.

It will be seen that, starting from A, the beam is first deflected with the concave part facing downwards, up to some point C. After this, as far as the centre, the concave part is upwards. So that at C there is a change in the direction of curvature. The two points in the beam where this occurs, C and D, are called points of *contrary flexure.*

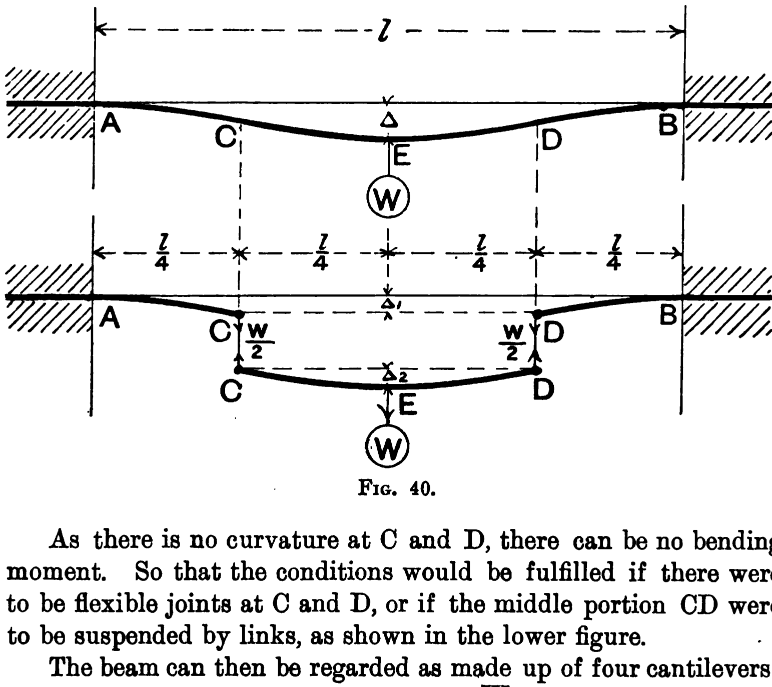

FIG. 40.

As there is no curvature at C and D, there can be no bending moment. So that the conditions would be fulfilled if there were to be flexible joints at C and D, or if the middle portion CD were to be suspended by links, as shown in the lower figure.

The beam can then be regarded as made up of four cantilevers: AC, EC, BD, and ED. The same force $\frac{W}{2}$ pulls down on the end of AC as pulls upward at the end of EC.

As the change of curvature is gradual at C and D, the angle of slope of the end C of AC must be the same as that of C on EC.

The four cantilevers are then in all respects similar, and their deflections Δ_1 and Δ_2 must be equal.

The required deflection of the beam at its centre will be made up of these two added together, or,

$$\triangle = \triangle_1 + \triangle_2 = 2\triangle_1$$

$$= 2\frac{\frac{W}{2}\left(\frac{l}{4}\right)^3}{3\,E\,I}$$

$$= \frac{Wl^3}{192\,E\,I}$$

Second Method.—The second way of treating beam deflections is more general than the first, and is based upon the assumption that for a curve whose abscissæ and ordinates are respectively x and y, the second differential coefficient, $\frac{d^2y}{dx^2} = \frac{1}{R}$ where R is the radius of curvature at the point xy. This is a very slightly modified form of a well-known mathematical result.

It has already been shown that $\frac{1}{R} = \frac{M}{EI}$, so that the above may be written

$$\frac{d^2y}{dx^2} = \frac{M}{E\,I} \quad \text{or} \quad E\,I\frac{d^2y}{dx^2} = M$$

CASE I. *Cantilever of length l, carrying a single load* W *at its outer end* (Fig. 41).

The bending moment at any point,

$$M = -W\,(l - x)$$

First, $$E\,I\frac{d^2y}{dx^2} = M = -W\,(l - x)$$

Now integrate $$E\,I\frac{dy}{dx} = -W\,l\,x + \frac{Wx^2}{2} + C \text{ (a constant)};$$

$\frac{dy}{dx}$ is the tangent of the angle of slope of the curve at X. When $x = 0$, $\frac{dy}{dx} = 0$, as the line is horizontal, and therefore C must $= 0$ also; when $x = l$, at the outer end, $E\,I\frac{dy}{dx} = -\frac{W\,l^2}{2}$

Again integrating, $$\mathrm{E\,I}\,y = -\frac{\mathrm{W}\,lx^2}{2} + \frac{\mathrm{W}x^3}{6} + \mathrm{B}\text{ (a constant)};$$

y is the vertical distance of any point below the horizontal through the support. When $x=0$, $y=0$, and therefore $\mathrm{B}=0$, so that

$$y = -\frac{\mathrm{W}lx^2}{2\,\mathrm{E\,I}} + \frac{\mathrm{W}x^3}{6\,\mathrm{E\,I}}$$

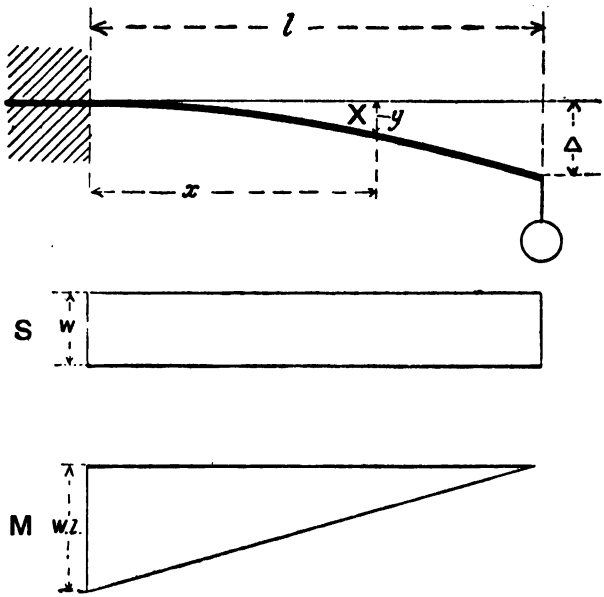

FIG. 41.

The deflection at the outer end when $x=l$ is

$$y = \triangle = \frac{\mathrm{W}l^3}{\mathrm{E\,I}}\left(-\frac{1}{2}+\frac{1}{6}\right)$$
$$= -\frac{\mathrm{W}l^3}{3\,\mathrm{E\,I}}$$

which is the same as was found by the first method.

In this and the following cases the diagrams showing the variation of the shearing force S, and the bending moment M, are drawn below the principal figure.

CASE II. *Cantilever of length l, carrying a uniformly distributed load w per foot* (Fig. 42).

Here the total load is $W = w\,l$, and the bending moment at any point X is

$$M = -w(l-x)\left(\frac{l-x}{2}\right) = -\frac{w\,l^2}{2} + w\,l\,x - \frac{wx^2}{2}$$

First,

$$E\,I\,\frac{d^2y}{dx^2} = M$$

$$= -\frac{w\,l^2}{2} + w\,l\,x - \frac{w\,x^2}{2}$$

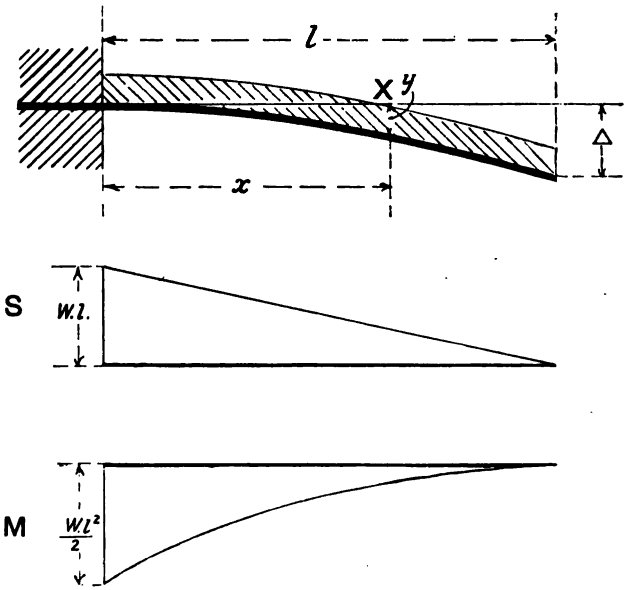

FIG. 42.

Integrating, this gives

$$E\,I\,\frac{dy}{dx} = -\frac{w\,l^2\,x}{2} + \frac{w\,l\,x^2}{2} - \frac{w\,x^3}{6} + C \text{ (a constant)};$$

when $x = 0$, $\frac{dy}{dx} = 0$, so that $C = 0$.

Integrating again,

$$E\,I\,y = -\frac{w\,l^2\,x^2}{4} + \frac{w\,l\,x^3}{6} - \frac{w\,x^4}{24} + B \text{ (a constant)};$$

when $x = 0$, $y = 0$, so that $B = 0$.

At the extreme outer end of the cantilever, when $x = l$, the slope is $-\frac{w\,l^3}{6\,\mathrm{E\,I}}$, and the deflection is

$$y = \Delta = -\frac{w\,l^4}{\mathrm{E\,I}}\left(\frac{1}{4} - \frac{1}{6} + \frac{1}{24}\right)$$

$$= -\frac{w\,l^4}{8\,\mathrm{E\,I}}$$

or, in terms of the total load,

$$\Delta = -\frac{\mathrm{W}l^3}{8\,\mathrm{E\,I}}$$

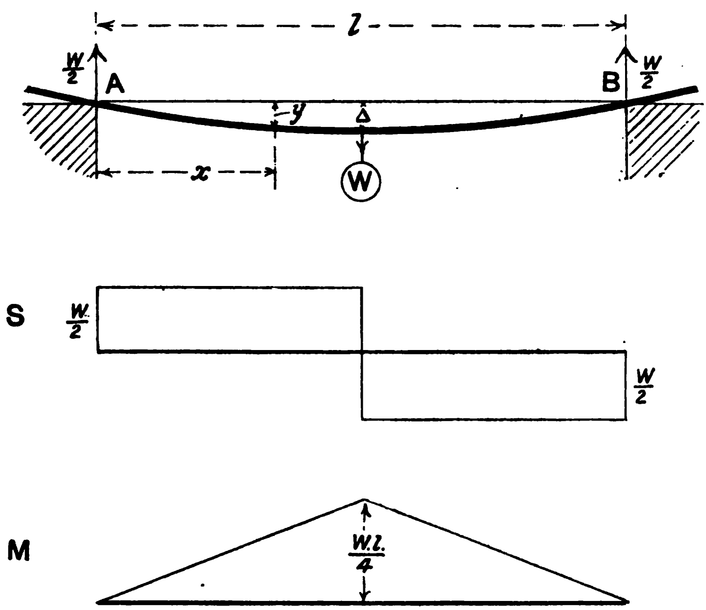

FIG. 43.

CASE III. *Beam supported at ends of span l, and carrying a concentrated load* W *in the centre* (Fig. 43).

Measuring x from the left-hand support A, the bending moment at any point x is $\mathrm{M} = +\frac{\mathrm{W}}{2}x$; and at the centre is $\frac{\mathrm{W}l}{4}$

Now, $$\mathrm{E\,I}\frac{d^2y}{dx^2} = \mathrm{M} = \frac{\mathrm{W}\,x}{2}$$

Integrating, $$E\,I\frac{dy}{dx} = \frac{Wx^2}{4} + C \text{ (a constant).}$$

When $$x = \frac{l}{2},\ \frac{dy}{dx} = 0$$

therefore, $$0 = \frac{W\,l^2}{16} + C$$

or $$C = -\frac{W\,l^2}{16}$$

Then $$E\,I\frac{dy}{dx} = \frac{Wx^2}{4} - \frac{Wl^2}{16}$$

Integrating again,

$$E\,I\,y = \frac{Wx^3}{12} - \frac{Wl^2x}{16} + B \text{ (a constant).}$$

When $x=0$, $y=0$, and therefore $B=0$.

Then the deflection at any point x is

$$y = \frac{Wx^3}{12\,E\,I} - \frac{W\,l^2x}{16\,E\,I}$$

The deflection at the centre, when $x=\frac{l}{2}$, is

$$\triangle = \frac{W\,l^3}{96\,E\,I} - \frac{W\,l^3}{32\,E\,I}$$
$$= -\frac{Wl^3}{48\,E\,I}$$

the *minus* sign again indicating that the deflection is measured downwards.

The slope $\frac{dy}{dx}$ is for that portion of the beam between the left-hand support and the centre. Beyond the centre the value of M is different owing to the presence of W.

At the left-hand support $x=0$, and the slope

$$i = \frac{dy}{dx} = \frac{W\,x^2}{4\,E\,I} - \frac{Wl^2}{16\,E\,I}$$
$$= -\frac{Wl^2}{16\,E\,I}$$

From the symmetry of loading, the numerical value of i at the other support is the same, but the sign is changed—that is,

$$i = \frac{W\,l^2}{16\,E\,I}$$

CASE IV. *Beam supported at ends of span l, and uniformly loaded with w per foot run* (Fig. 44).

Total load on beam $= W = wl$

Reactions at supports, $R_1 = R_2 = \frac{w\,l}{2}$

Bending moment at any point X is

$$M = \left(\frac{wl}{2}\right)x - \left(w\,x\right)\frac{x}{2} = \frac{w\,l\,x}{2} - \frac{w\,x^2}{2}$$

As before, $$E\,I\frac{d^2y}{dx^2} = M = \frac{wlx}{2} - \frac{w\,x^2}{2}$$

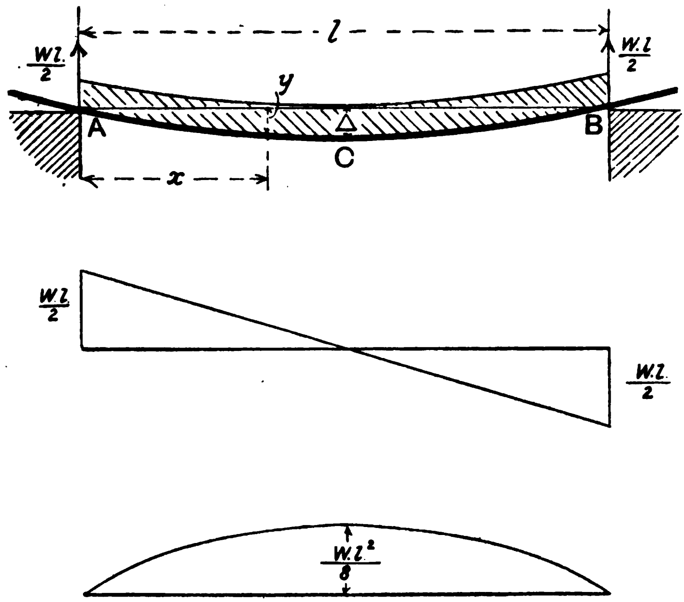

FIG. 44.

Integrating, $$E\,I\frac{dy}{dx} = \frac{w\,l\,x^2}{4} - \frac{w\,x^3}{6} + C \text{ (a constant)},$$

when $$x = \frac{l}{2},\ \frac{dy}{dx} = 0,$$

therefore, $$0 = \frac{w\,l^3}{16} - \frac{w\,l^3}{48} + C$$

or $$C = -\frac{wl^3}{24}$$

so that
$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} - \frac{wl^3}{24}$$

Integrating again,

$$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24} + B \text{ (a constant)}$$

when $x=0$, $y=0$, and therefore $B=0$, so that the deflection at X is

$$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24}$$

At the centre, where $x = \frac{l}{2}$, the deflection is

$$y = \Delta = \frac{wl^4}{96\,EI} - \frac{wl^4}{384\,EI} - \frac{wl^4}{48\,EI}$$
$$= -\frac{5wl^4}{384\,EI}$$
$$= -\frac{5}{384}\ \frac{Wl^3}{EI}$$

The slope at any point X is

$$i = \frac{dy}{dx} = \frac{wlx^2}{4\,EI} - \frac{wx^3}{6\,EI} - \frac{wl^3}{24\,EI}$$

at the left-hand support this is

$$i = -\frac{wl^3}{24\,EI}$$

CASE V. *Beam fixed at ends of span l, and carrying a concentrated load* W *in the centre* (Fig. 45).

It is assumed in this case that (*a*) the two ends of the free part A and B are in the same horizontal line, and (*b*) that the ends are so restrained that the neutral surface is horizontal at these points, and consequently the tangents of the angles of slope, $\frac{dy}{dx} = 0$.

First, suppose the beam to rest freely on the supports, as in Case III. The slope at the supports is then $= \frac{Wl^2}{16\,EI}$.

A bending moment M_1 will now have to be applied at each support, in order to bend the overhanging ends downwards, as indicated by the arrows, until $\frac{dy}{dx} = 0$.

Neglecting this moment, the reactions at A and B caused by W are $=\frac{W}{2}$. The bending moment at any point of the beam X is $M = \frac{Wx}{2} - M_1$ when M_1 is the moment necessary to keep the neutral surface horizontal at the supports. Every part of the beam is subjected to this same moment.

To find M_1 $$EI\frac{d^2y}{dx^2} = M = \frac{Wx}{2} - M_1$$

Integrating, $$EI\frac{dy}{dx} = \frac{Wx^2}{4} - M_1 x + C$$

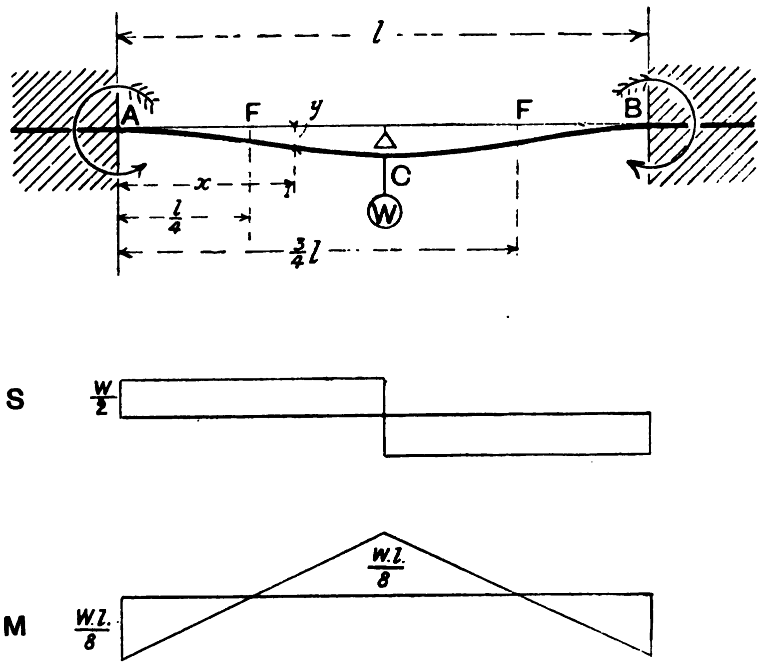

FIG. 45.

At the supports, where $x=0$, $\frac{dy}{dx}=0$; $\therefore C=0$; when $x=\frac{l}{2}$ $\frac{dy}{dx}$ again $=0$, and $$0 = \frac{Wl^2}{16} - M_1\frac{l}{2}$$

or $$M_1 = \frac{Wl}{8}$$

Then
$$E\,I\frac{dy}{dx} = \frac{Wx^2}{4} - \frac{Wlx}{8}$$

$$E\,I\,y = \frac{Wx^3}{12} - \frac{Wlx^2}{16} + B$$

When $x=0,\ y=0,$ and $B=0$

When
$$x = \frac{l}{2},\ y = \Delta = \left(\frac{Wl^3}{96} - \frac{Wl^3}{64}\right)\frac{1}{E\,I}$$

$$= -\frac{Wl^3}{192\,E\,I}$$

The deflection of any other point at x is

$$y = \frac{1}{E\,I}\left(\frac{Wx^3}{12} - \frac{Wl\,x^2}{16}\right)$$

The points F F are points of *contrary flexure*, where the curvature changes and there is no bending moment. To find these, put

$$M = 0$$

$$\therefore\quad 0 = \frac{W\,x}{2} - \frac{Wl}{8}$$

from which
$$x = \frac{l}{4}$$

The slope at any part is
$$i = \frac{dy}{dx} = \frac{W}{E\,I}\left(\frac{x^2}{4} - \frac{lx}{8}\right)$$

CASE VI. *Beam of span l fixed at the supports and uniformly loaded with w per foot run* (Fig. 46).

The conditions of supporting the beam are precisely similar to those in the last case. The total load is $wl = W$, and the reactions at the supports due to this alone are $= \frac{W}{2} = \frac{w\,l}{2}$

The bending moment at any point X is

$$M = \frac{wlx}{2} - \frac{w\,x^2}{2} - M_1$$

$$E\,I\frac{d^2y}{dx^2} = M = \frac{w\,l\,x}{2} - \frac{wx^2}{2} - M_1$$

$$E\,I\frac{dy}{dx} = \frac{w\,l\,x^2}{4} - \frac{wx^3}{6} - M_1x + C$$

At the support, where $x=0$, $\frac{dy}{dx}=0$; $\therefore$ C = 0, and, at the centre, $\frac{dy}{dx}=0$; and $x=\frac{l}{2}$, that is

$$0 = \frac{wl^3}{16} - \frac{w\,l^3}{48} - \mathrm{M}_1\,\frac{l}{2}$$

from which

$$\mathrm{M}_1 = \frac{wl^2}{12}$$

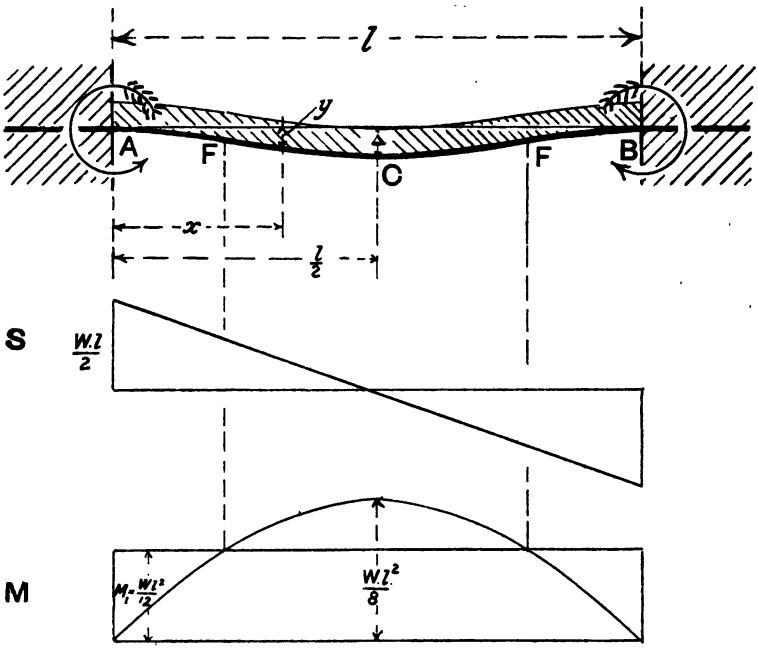

FIG. 46.

Then

$$\mathrm{E\,I}\frac{dy}{dx} = \frac{wl\,x^2}{4} - \frac{w\,x^3}{6} - \frac{wl^2x}{12}$$

$$\mathrm{E\,I}\,y = \frac{wlx^3}{12} - \frac{w\,x^4}{24} - \frac{wl^2x^2}{24} + \mathrm{B}$$

when $x=0$, $y=0$; $\therefore$ B = 0, and when $x=\frac{l}{2}$

$$y = \Delta = -\frac{w\,l^4}{384\,\mathrm{E\,I}}$$

$$= -\frac{\mathrm{W}l^3}{384\,\mathrm{E\,I}}$$

The slope at any point of this beam is

$$i = \frac{dy}{dx} = \frac{1}{\mathrm{E\,I}}\left(\frac{w\,l\,x^2}{4} - \frac{w\,x^3}{6} - \frac{w\,l^2\,x}{12}\right)$$

There are two points of *contrary flexure,* where the bending moment is zero, one on each side the centre. To find the positions of these, put

$$\mathrm{M} = 0 = \frac{w\,l\,x}{2} - \frac{w\,x^2}{2} - \frac{wl^2}{12}$$

from which

$$x = \frac{l}{2} \pm \frac{l}{2\sqrt{3}}$$

which are the distances of the points FF from A.

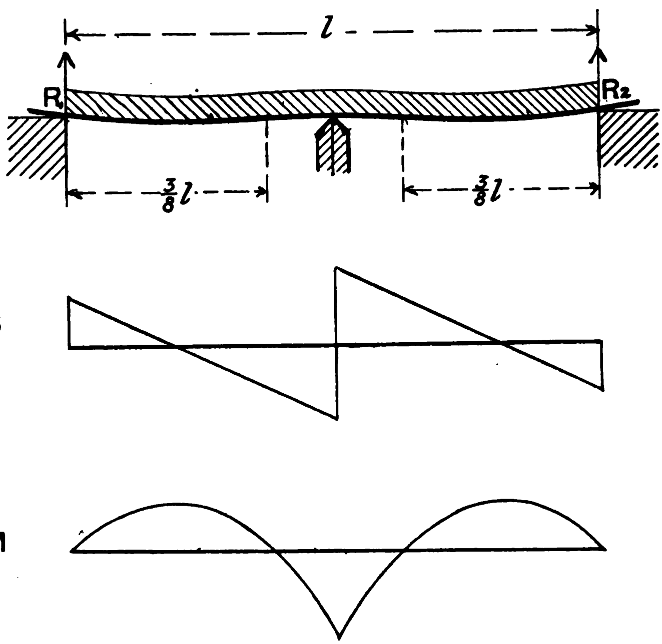

Fig. 47.

To find the reactions at the supports in the case of a beam uniformly loaded and resting upon three supports, equally spaced and in the same horizontal plane (Fig. 47).

This problem is best solved by making use of the fact that "the deflection of a beam acted upon by several loads is the sum of the deflections due to the several loads."

First, suppose the centre support to be taken away and the beam to rest freely on the remaining supports, whose distance apart is l. Then, if the total load is W, the deflection in the centre is

$$\triangle = \frac{5}{384}\frac{Wl^3}{EI}$$

Now, raise the central support until the centre is restored to the level of the outer supports. Call the force required to effect this P. The case now resolves itself into an upward load P applied in the centre of a freely supported beam of l span, and producing a central deflection $= \triangle$.

Considered from this point of view,

$$\triangle = \frac{Pl^3}{48\,EI}$$

Also,

$$\triangle = \frac{5}{384}\frac{Wl^2}{EI}$$

so that

$$\frac{Pl^3}{48\,EI} = \frac{5}{384}\frac{Wl^3}{EI}$$

and

$$P = \frac{240}{384}W$$

$$= \frac{5}{8}W$$

Also,

$$R_1 = R_2 = \frac{W}{2}\left(1 - \frac{5}{8}\right)$$

$$= \frac{3}{16}W$$

Case of a cantilever uniformly loaded, with the outer end resting on a support at the same level as the fixed end (Fig. 48).

This may be treated in a manner similar to that employed in the last case. By equating the (downward deflection caused by the uniform load acting on the free cantilever) to the (upward deflection caused by a load at the outer end), this isolated load, which is the reaction on the support, is found to be 3/8 wl, or 3/8 W.

The point of contrary flexure is 3/4 l from the outer end, or 1/4 l from the fixed end.

Example 1.—In a rolled steel beam (symmetrical about the neutral axis), the moment of inertia of the section is 72 inch-units.

The beam is 8 ins. deep, is laid across an opening of 10 ft., and carries a distributed load of 9 tons. Find the maximum fibre stress, also the central deflection, taking E at 13,000 tons per sq. in.

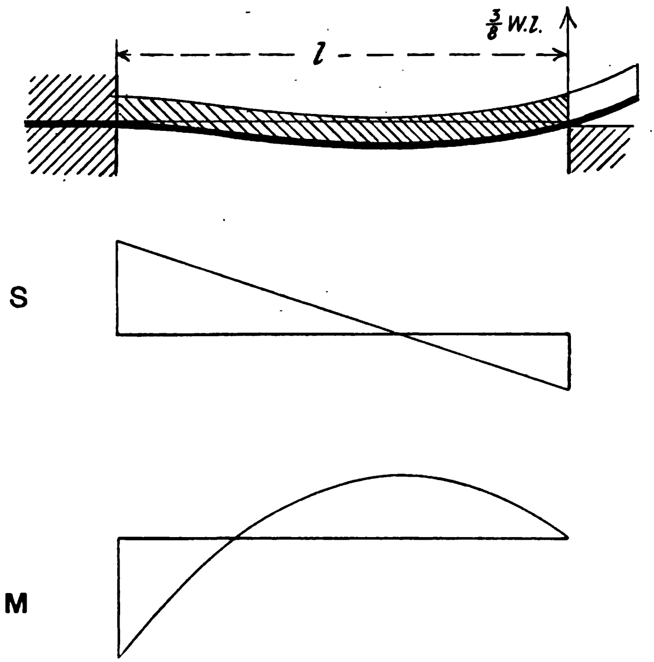

Fig. 48.

Let $\triangle$ = deflection in inches;
M = bending moment in inch-tons (maximum bending moment);
f = extreme fibre stress in tons per sq. in.;
W = total distributed load in tons;
l = width of opening in inches;
y = distance from neutral axis to extreme edge of section in inches;
I = moment of inertia of section in inch-units.

The maximum bending moment occurs at the centre, and is

$$M = \frac{Wl}{8} = \frac{9 \times 10 \times 12}{8}$$
$$= 135 \text{ inch-tons.}$$

The maximum fibre stress,

$$f = \frac{My}{I} = \frac{135 \times 4}{72}$$
$$= 7 \cdot 5 \text{ tons per sq. in.}$$

The central deflection,

$$\triangle = \frac{5}{384} \frac{Wl^3}{EI}$$
$$= \frac{5 \times 9 \times 120 \times 120 \times 120}{384 \times 13000 \times 72}$$
$$= 0 \cdot 216 \text{ ins.}$$

Example 2 (I.C.E.).—Suppose that three beams or planks, A, B, and C, of the same material, are laid side by side across a span $l = 100$ ins., and a load $W = 600$ lbs. is laid across them at the centre of the span, so that they must all bend together. The beams are all 6 ins. wide, but while A and C have a depth of 3 ins., the depth of the middle beam B is twice as great. How much of the weight W will be carried by each of the three beams, and what will be the extreme fibre stress in each?

Let l = length of beams in inches;
E = modulus;
I_1 = moment of inertia of $6'' \times 6''$ beam in inch-units;
I_2 = moment of inertia of two $6'' \times 3''$ beams in inch-units;
$\triangle$ = deflection of beams in inches;
M_1 = maximum bending moment of $6'' \times 6''$ beam in inch-lbs.;
M_2 = maximum bending moment of two $6'' \times 3''$ beams in inch-lbs.;
y = distance from neutral axis to extreme edge of section;
f = extreme fibre stress in lbs. per sq. in.

Suppose the two $6'' \times 3''$ beams to be taken as *one* $12'' \times 3''$ beam.

The load being placed across all three beams, and all bending together, the deflection will be the same in each case.

Let x = the load carried by the $6'' \times 6''$ beam.

Then $W - x$ = the load carried by the two $6'' \times 3''$ beams.

$$\text{Deflection} = \triangle = \frac{xl^3}{48 \, E \, I_1} = \frac{(W - x)l^3}{48 \, E \, I_2}$$

$$\text{or} \quad \frac{x}{I_1} = \frac{(w - x)}{I_2}$$

$$\text{But} \quad I_1 = \frac{bd^3}{12} = \frac{6 \times (6)^3}{12} = 108 \text{ inch-units}$$

$$\text{and} \quad I_2 = \frac{12\,(3)^3}{12} = 27 \text{ inch-units.}$$

$$\text{Therefore} \quad \frac{x}{108} = \frac{(600 - x)}{27}$$

$$x = \frac{108\,(600 - x)}{27}$$

$$x = 480 \text{ lbs.}$$

$$600 - x = 120 \text{ lbs.}$$

$$\text{and} \quad \tfrac{1}{2} \times (600 - x) = 60 \text{ lbs.}$$

Load carried by 6″ × 6″ beam is 480 lbs.
Load carried by each 6″ × 3″ beam is 60 lbs.
Bending moment for 6″ × 6″ beam

$$= M_1 = \frac{WL}{4} = \frac{480 \times 100}{4} = 12{,}000 \text{ inch-lbs.}$$

Bending moment for 6″ × 3″ beam (each)

$$= M_2 = \frac{WL}{4} = \frac{60 \times 100}{4} = 1500 \text{ inch-lbs.}$$

Maximum fibre stress for 6″ × 6″ beam

$$= f_1 = \frac{M_1 y_1}{I_1} = \frac{12000 \times 3}{108}$$

$$= 333{\cdot}\dot{3} \text{ lbs. per sq. in.}$$

$$\text{and} \quad f_2 = \frac{M_2 y_2}{I_2} = \frac{3000 \times \frac{3}{2}}{27} = 166{\cdot}\dot{6} \text{ lbs. per sq. in.}$$

CHAPTER VI

SHEAR STRESS IN LOADED BEAMS

Distribution of Shear Stress in Loaded Beams

It is not difficult to see that there must be shear stress produced on vertical sections of horizontal loaded beams by the shearing forces which tend to cause sliding. The distribution of this stress over the section will now be considered.

If there is shear stress at any point on a vertical section, it has been shown that there must also be a shear stress of equal intensity on a horizontal plane going through the same point. It is this horizontal stress that is first determined, and the shear stress on a vertical section taken as being equal.

Fig. 49.

The simplest illustration of the existence of a horizontal shear stress is to be found in the case of a rectangular timber beam which has been sawn into planks, as shown in Fig. 49. On these being supported at the ends, it is seen that they are deflected downwards by their own weight to a very much greater extent than would be the case with the original solid beam. The reason is that the horizontal layers are free to slide upon one another, and become so many individual beams. In the original beam the tendency to slide in this way is resisted by the horizontal shear stress in the material.

To find the actual intensity of the shear stress f_s at any point

whose distance from the neutral axis of the vertical section is r, the following method may be used:

Referring to Fig. 50, consider two vertical sections AA and BB, enclosing a slice of the material of the beam in question, and whose distance apart is δz. Next, consider the equilibrium of that portion of the slice which lies above any horizontal plane LL_1, distant r from the neutral axis. This portion is acted upon by horizontal forces only, and is in equilibrium. Three forces act upon it, namely, the horizontal direct stress on $AL = P$; the contrary horizontal stress on $BL_1 = P_1$; and the shear force on $LL_1 = F$. Let the former of these direct forces be the greater.

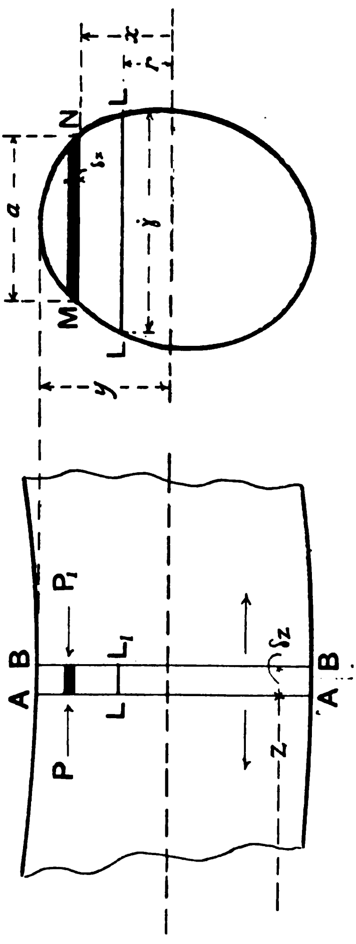

FIG. 50.

Using the same notation as before,

$$P = \int_r^y f_A \frac{x}{y} a\, dx$$

Also $$P_1 = \int_r^y f_B \frac{x}{y} a\, dx$$

where f_A and f_B are the maximum direct stresses at A and B, and $F = f_s j\, \delta z$, f_s being the required shear stress, j the length, and δz the width of the shear surface. But, if A is the vertical area above LL, and X the distance of its centre of gravity from the neutral axis,

$$AX = \int_r^y x\, a\, d\, x$$

Therefore $$P = \frac{f_A}{y} AX$$

$$\text{and} \qquad P_1 = \frac{f_B}{y} AX$$

Also, $F = P - P_1$, and, in the limit

$$f_s\, j\, dz = \frac{AX}{y}(f_A - f_B)$$

$$dz = \frac{AX}{f_s\, j\, y}(f_A - f_B)$$

If M is the bending moment at AA, and $M - dM$ the bending moment at BB,

$$M = f_A \frac{I}{y}, \text{ and } M - dM = f_B \frac{I}{y},$$

from which $$dM = \frac{I}{y}(f_A - f_B)$$

Therefore $$\frac{dM}{dz} = \frac{I f_s j}{AX}$$

But $\frac{dM}{dz}$ is the shearing force S at the section, from which it follows that

$$f_s = \frac{SAX}{Ij}$$

which is the shear stress required.

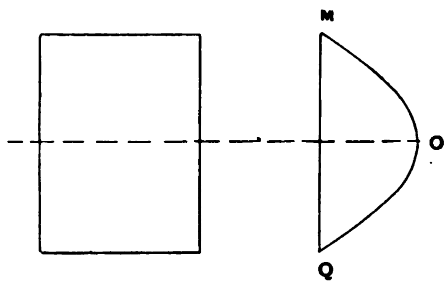

FIG. 51.

Example.—To find the distribution of shear stress on the vertical section of a rectangular beam of width b and depth $d = 2h$, when subjected to a total shearing force S (Fig. 51).

Using the value found above,

$$f_s = \frac{SAX}{Ij},$$

A is the area of the section above LL (Fig. 50), $= b(h - r)$

X is the distance of its C.G. from the neutral axis, $= \frac{h + r}{2}$

I is the moment of inertia of the whole section, $= \frac{bh^3 8}{12}$

j is constant and $= b$.

Putting these values in the above equation, the required stress,

$$f_s = \frac{3}{4} S \frac{(h^2 - r^2)}{h^3 b}$$

$$= 6 S \frac{\left(\frac{d^2}{4} - r^2\right)}{d^3 b}$$

If the values of f_s are found for a number of different values of r, and the results plotted from a vertical base-line, the variation of f_s will be shown by the parabolic curve MOQ.

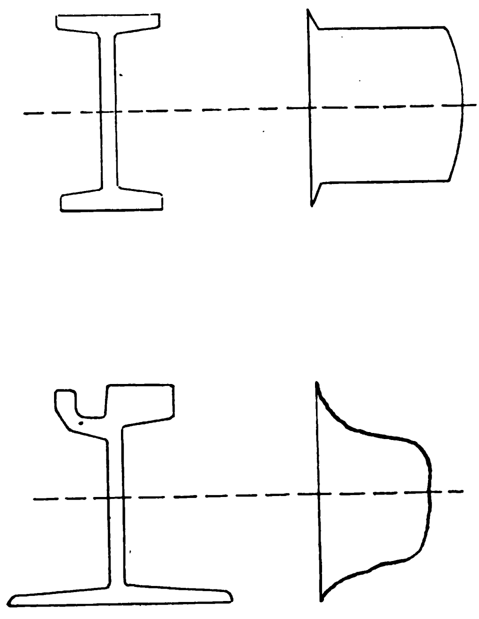

Fig. 52.

On Fig. 52 are shown curves of shear stress for a beam of I section and for a tram-rail.

Deflection Due to Shear.

The way in which the deflection caused by the bending moment on a beam is found was discussed in the last chapter. In addition to the deflection so found, there is an additional

amount to be added, caused by the effect of the shearing forces in producing shear distortion.

In beams where the ratio $\frac{\text{span}}{\text{depth}}$ is great, the deflection due to shear is so small as to be negligible; but in short, deep beams it becomes of more importance.

Professor Perry has shown that for a rectangular cantilever loaded at the end only, the total deflection at the end is

$$\triangle_1 = \frac{Wl^3}{3EI} + \frac{6}{5}\,\frac{Wl}{Gbd}$$

and of the two parts which go to make up this total, that on the left is the one already found; that on the right gives the deflection due to shear. Here G is the shear modulus.

Continuous Beams.

Let a continuous beam rest freely on supports which are on the same level at A, B, C, D (Fig. 53). At the first and last

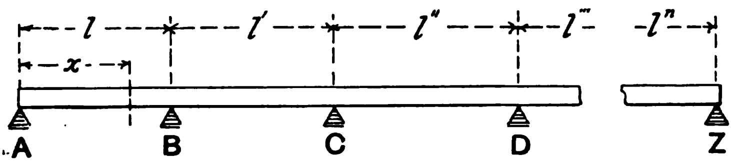

FIG. 53.

support, A and Z, there will be no bending moment; at each of the others there will be a bending moment due to the difference in inclination of the tangents at A and the other supports.

Let M_A, R_A, z_A, and M_B, R_B, z_B, etc., be the respective bending moments, reactions, and angles of slope at A, B, etc.

First, let the beam carry uniform loads per ft. run, w, w', w'', etc., in the respective spans l, l', l''.

The bending moment at a point distant x from A is

$$M = M_A + \frac{wx^2}{2} - R_A x$$

and

$$M_B = M_A + \frac{wl^2}{2} - R_A l$$

from which

$$R_A = \frac{M_A - M_B}{l} + \frac{wl}{2} \qquad (1)$$

Therefore
$$M = M_A + \frac{wx^2}{2} - (M_A - M_B)\frac{x}{l} - \frac{wlx}{2}$$
$$= EI\frac{d^2y}{dx^2}$$

Whence, on integrating,
$$EI\frac{dy}{dx} = M_A x + \frac{wx^3}{6} - \frac{M_A - M_B}{2}\frac{x^2}{l} - \frac{wlx^2}{4} + C \qquad (2)$$

when $x=0$, $\frac{dy}{dx}=z_A$, so that $C=EI\,z_A$; when $x=l$, $\frac{dy}{dx}=z_B$

Therefore
$$EI\,z_B = M_A l + \frac{wl^3}{6} - (M_A - M_B)\frac{l}{2} - \frac{wl^3}{4} + EI\,z_A$$
$$= (M_A + M_B)\frac{l}{2} - \frac{w\,l^3}{12} + EI\,z_A \qquad (3)$$

Again, integrating (2),
$$EIy = \frac{M_A x^2}{2} + \frac{wx^4}{24} - (M_A - M_B)\frac{x^3}{6l} - \frac{wlx^3}{12} + EI\,z_A x$$

when $x=0$, $y=0$, and when $x=l$, $y=0$, so that the constant $=0$, that is
$$0 = \frac{M_A l^2}{2} + \frac{wl^4}{24} - (M_A - M_B)\frac{l^2}{6} - \frac{wl^4}{12} + EI\,z_A l$$

or
$$EIz_A = -M_A\frac{l}{3} - M_B\frac{l}{6} + \frac{wl^3}{24}$$

In a similar manner, taking B as origin and considering the span $BC = l'$, the following result is obtained:—
$$EI\,z_B = -M_B\frac{l'}{3} - M_C\frac{l'}{6} + \frac{w'l'^3}{24}$$
$$= (M_A + M_B)\frac{l}{2} - \frac{wl^2}{12} + EI\,z_A$$
$$= (M_A + M_B)\frac{l}{2} - \frac{wl^2}{12} - M_A\frac{l}{3} - M_B\frac{l}{6} + \frac{wl^3}{24}$$
$$= M_A\frac{l}{6} + M_B\frac{l}{3} - \frac{wl^3}{24}$$

Therefore
$$M_A\frac{l}{6} + \frac{M_B}{3}(l+l') + \frac{M_C l'}{6} = \frac{(wl^2 + w'l'^3}{24}$$

From which

$$M_A l + 2M_B(l+l') + M_C l' = \frac{wl^3 + w'l'^3}{4} \qquad \text{(i)}$$

This last is the *equation of* 3 *moments*, and is similar for each pair of spans. That is,

$$M_B l' + 2M_C(l' + l'') + M_D l'' = \frac{w' l'^3 + w'' l''^3}{4} \qquad \text{(ii)}$$

$$M_C l'' + 2M_D(l'' + l''') + M_E l''' = \frac{w'' l''^3 + w''' l'''^3}{4} \qquad \text{(iii)}$$

and so on.

There are five unknown quantities in the above three equations (i), (ii), and (iii). It generally happens that the beam is allowed to rest freely on the supports at the two extreme ends, so that $M_A = 0$ and $M_B = 0$, and there will be three unknowns to be found from three equations, so that these values are determinate.

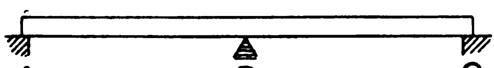

FIG. 54.

Example 1.—Case of uniform load on a continuous beam resting on three supports, separated by two equal spans (Fig. 54).

Here, using equation (i), $M_A l + 2M_B(l + l') + M_C l' = \frac{wl^3 + w'l'^3}{4}$

$$M_A = 0, \ M_C = 0, \ w = w', \text{ and } l = l'$$

so that

$$M_B = \frac{wl^2}{8}$$

and

$$R_A l - \frac{wl^2}{2} = \frac{wl^2}{8}$$

or

$$R_A = \frac{3}{8} wl = R_C \text{ by reason of symmetry.}$$

Total load $= 2wl$, so that

$$R_B = 2wl - 2\frac{3}{8}wl = \frac{5}{4} wl$$

calling the total load

$$W = 2wl$$

$$R_A = R_C = \frac{3}{16} W$$

and

$$R_B = \frac{5}{8} W$$

This is the same result as was previously obtained in another way.

Example 2.—Case of uniform load in a continuous beam resting on four supports, separated by three equal spans (Fig. 55).

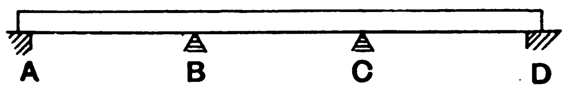

FIG. 55.

Using equations (i), (ii), and (iii)—

M_A and $M_D = 0$, $w = w' = w''$, and also $M_B = M_C$, $l = l' = l''$,

so that it is only necessary to write (i) as

$$4M_B l + M_C l = \frac{wl^3}{2}$$

which becomes

$$5M_B l = \frac{wl^3}{2}$$

or

$$M_B = \frac{wl^2}{10} = M_C$$

and

$$R_A = \frac{M_A - M_B}{l} + \frac{wl}{2}$$

$$= -\frac{wl}{10} + \frac{wl}{2} = \frac{2}{5} wl$$

and from symmetry

$$R_0 = R_A = \frac{2}{5} wl$$

The total load on the beam is $3wl$, so that

$$R_B = R_C = \frac{1}{2}\left(3wl - \frac{4}{5}wl\right) = \frac{11}{10}\,wl$$

or if the total load $3wl$ is called W,

$$R_A = R_O = \frac{4}{30}\,W$$

and

$$R_B = R_C = \frac{11}{30}\,W$$

CHAPTER VII

RELATION BETWEEN LOAD AND STRESS IN A PRISMATIC BAR

REFERRING to the accompanying figure (Fig. 56), the upper view represents the elevation of a prismatic bar or piece of material, *l, m, o, p,* whose cross-section on a plane at right angles to the axis is shown in the lower view. The geometric axis of the bar will pass through the centre of gravity of this section. Let a load P act either along or parallel to this axis, as shown by the dotted line. This may be either a compressive or a tensile load. Suppose for the present that it is a load in compression. If it acts along the axis of the piece, it will result in a uniform compressive stress which is the same at all points of the section. Its magnitude will be $f_d = \frac{P}{A}$, where A is the area in question.

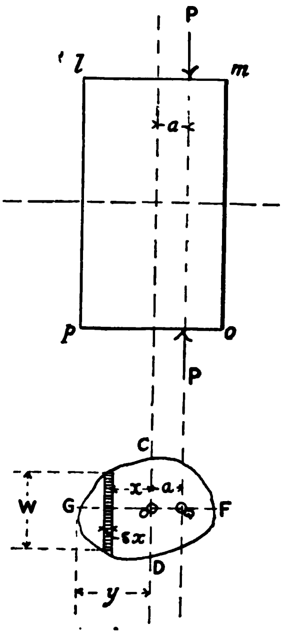

FIG. 56.

In most cases of loading of this kind, uniformity of stress is aimed at but is rarely attained. If the line of the load P is shifted ever so little away from the geometric axis of the prism, the stress at once becomes a variable one, with a maximum value greater and a minimum less than f_d. The farther the load is moved away from the axis, the greater becomes the difference between the mean and the extreme stresses. The precise relation which exists between the load and the extreme and mean stresses for any given section, when the line

of loading is so moved away from the geometric axis, is given in what immediately follows.

Again referring to Fig. 56, the load P is supposed no longer to act at the centre of gravity of the area O, but its line of action, still parallel to the axis, has been moved away from this point, and now passes through some new point, Q, at a distance a from O measured along FG, one of the principal axes of the figure. The point Q may or may not lie in this line, but for the present it is to be supposed that it does.

It has been shown that if M is the bending moment on a section of a prism, $M=\frac{f_b I}{y}$, where I is the moment of inertia of the area about the neutral axis CD. In the present case the load P is one of compression, and, in acting as it does at some distance from the centroid of the area, gives rise to the bending moment M. This tends to bend the piece in the direction indicated by the arrows, causing compression of the material at F and tension at G. Both stresses have a maximum value at these points, and diminish uniformly as they approach the neutral axis. If the section is symmetrical about the neutral axis and the elastic properties are the same in compression as in tension, it has been shown that the value of the maximum stress, $f_b = \frac{My}{I}$, will be the same at F as at G. The precise value of the bending moment will in the present case be $M=Pa$, and therefore $f_b=\frac{Pay}{I}$. This gives the value of the maximum tensile and compressive stresses due to the bending action caused by the load in acting away from the centroid of the area.

It has just been shown that the uniform stress is the load divided by the area of the section, or $f_d=\frac{P}{A}$.

The actual combined stress at either edge, F or G, will be the algebraic sum of the uniform and bending stresses.

Thus, at F, which is on the same side of the neutral axis as the point where the load acts, the two stresses will have the same sign. In the present instance this is one of compression. If the uniform stress f_d be given the plus sign, then the stress due to bending,

$$f_b = \pm\frac{My}{I}$$

according as it is on the same or the opposite side of the neutral axis as the load P. At any point in the section there will be a combined stress, consisting on the one part of the uniform stress and on the other of the stress due to the bending.

At F, the total or effective stress will be

$$f_e = f_d + f_b = \frac{P}{A} + \frac{My}{I}$$

and at G,

$$f_e = f_d - f_b = \frac{P}{A} - \frac{My}{I}$$

As the moment of inertia $I = A \cdot k^2$ where, as before, A is the area and k is the radius of gyration about CD, the above expression may for convenience be written

$$f_e = \frac{P}{A} \pm \frac{My}{Ak^2}$$

$$= \frac{P}{A}\left(1 \pm \frac{ay}{k^2}\right)$$

The above expression gives a relation between the dimensions of the section of the piece of material under stress, the magnitude of the load, the distance of the line of action of the load from the centre of gravity, and the maximum and minimum stresses resulting. It is applicable to sections of any form, so long as the moments of inertia can be found. In the case of the more usual sections, which consist wholly of, or are made up of regular figures, such as rectangles and circles, the values of the radius of gyration can readily be found by the usual mathematical methods; where the section has a more complicated form the modulus must be found by the graphical method, and from it the radius of gyration determined.

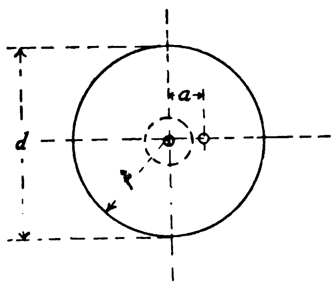

FIG. 57.

Particular Cases.—One or two particular cases will now be considered.

Circle.—In Fig. 57 is shown a circular section of radius r and diameter d. In this case the distance from the neutral axis or

centre of gravity to the edge of the section is the radius of the circle, and the expression giving the maximum stress is

$$f_e = \frac{P}{A}\left(1+\frac{ar}{k^2}\right)$$

and as in the circle

$$k^2 = \frac{r^2}{4}, \text{ or } \frac{d^2}{16}$$

the above becomes

$$f_e = \frac{P}{A}\left(1+\frac{4a}{r}\right)$$

By again subtracting the uniform stress resulting from the load acting down the geometrical axis of the bar from the maximum stress obtained as above, the stress due to bending alone, which may be called the excess caused by the eccentric loading, is obtained. That is,

$$\text{Excess} = e = f_e - f_d = \frac{P}{A}\left(1-\frac{4a}{r}\right) - \frac{P}{A} = f_b$$

which reduces to $\frac{P}{A}\frac{4a}{r}$; or, when put in the form of a percentage by which the maximum exceeds the uniform or mean, it becomes $\frac{4a100}{r}$ or $\frac{8a100}{d}$, where d is the diameter of the circle.

In the following table are given values of this percentage for different degrees of eccentricity, varying from 0 to 0·50d; that is to say, at various points from the centre to the circumference of the circle.

Table of Percentage Excess of Maximum over Uniform or Mean Stress in a Circular Section.

a	e	Remarks.
0	0	At centre
0·05d	40 per cent.	
0·10d	80 ,,	
0·20d	160 ,,	
0·30d	240 ,,	
0·40d	320 ,,	
0·50d	400 ,,	At circumference

Thus, it will be seen that by moving the line of action of the load no more than one-tenth of the diameter of the circle away from its centre, the stress at the circumference is about

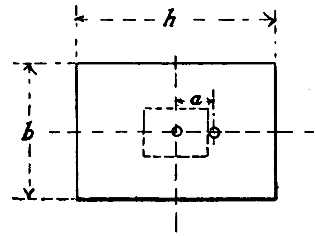

FIG. 58.

doubled, and by taking it to the circumference the stress is magnified fivefold.

Rectangle.—In the case of a rectangular section (Fig. 58) similar results are found. Here, as before,

$$f_b = e = \frac{ay}{k^2} 100 = \frac{6a}{h} 100$$

where a has the same meaning as before, and h is the depth of the section measured at right angles to the neutral axis. Here $k^2 = \frac{h^2}{12}$.

TABLE OF PERCENTAGE EXCESS OF MAXIMUM OVER UNIFORM OR MEAN STRESS IN A RECTANGULAR SECTION.

a	e	Remarks.
0	0	At centre
0·05h	30 per cent.	
0·10h	60 "	
0·20h	120 "	
0·30h	180 "	
0 40h	240 "	
0·50h	300 "	At edge

By comparing the figures in these two tables it will at once be seen that the increase is more marked in the case of the circle than in the rectangle.

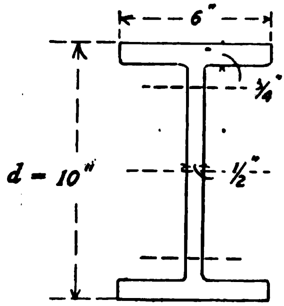

FIG. 59.

Example of Girder Section (Fig. 59):—

$$\text{Here A} = \begin{cases} 2 \times 6 \times \frac{3}{4} = 9 \\ 8\frac{1}{2} \times \frac{1}{2} = 4\frac{1}{4} \end{cases}$$

$$= \underline{13\tfrac{1}{4}} \text{ sq. ins.} \qquad y = 5 \text{ ins.}$$

$$k^2 = \frac{\mathrm{I}}{\mathrm{A}} = \frac{\dfrac{6 \times (10)^3}{12} - \dfrac{5\frac{1}{2} \times (8\frac{1}{2})^3}{12}}{13\frac{1}{4}} = \underline{16 \cdot 5}$$

$$e = \text{per cent. excess} = \frac{ay}{k^2} \times 100$$

when $a = 0 \cdot 1d$, $e = \dfrac{1 \times 5 \times 100}{16 \cdot 5} = 30$ per cent.

,, $a = 0 \cdot 2d$, $e = \dfrac{2 \times 5 \times 100}{16 \cdot 5} = 60$,,

,, $a = 0 \cdot 3d$, $e = \dfrac{3 \times 5 \times 100}{16 \cdot 5} = 90$,,

,, $a = 0 \cdot 4d$, $e = \dfrac{4 \times 5 \times 100}{16 \cdot 5} = 120$,,

,, $a = 0 \cdot 5d$, $e = \dfrac{5 \times 5 \times 100}{16 \cdot 5} = 150$,,

TABLE OF PERCENTAGE EXCESS OF MAXIMUM OVER UNIFORM OR MEAN STRESS IN THE I SECTION SHOWN ON FIG. 59.

a	e	Remarks.
0	0	At centre
$0{\cdot}10d$	30 per cent.	
$0{\cdot}20d$	60 ,,	
$0{\cdot}30d$	90 ,,	
$0{\cdot}40d$	120 ,,	
$0{\cdot}50d$	150 ,,	At edge

It will be seen that the more the material is spread away from the centre the smaller is the effect produced by moving the load-line any given distance from the neutral axis.

Position of the load P *which will give a zero stress at one edge of the section.*

The minimum stress is

$$= f_d - f_b$$

$$= \frac{\mathrm{P}}{\mathrm{A}}\left(1 - \frac{ay}{k^2}\right)$$

Put this equal to zero

$$0 = \frac{\mathrm{P}}{\mathrm{A}}\left(1 - \frac{ay}{k^2}\right)$$

from which $a = \frac{k^2}{y}$, which means that if P acts along a line which passes through the section at a distance $\frac{k^2}{y}$ from its centre of gravity measured at right angles to the neutral axis, the stress due to bending will just neutralise the direct stress at the edge of the section on the side farthest away from the load.

For a circle, where $k^2 = \frac{d^2}{16}$ and $y = \frac{d}{2}$

$$a = \frac{d^2 2}{16d} = \frac{d}{8}$$

That is to say, P must act within the middle quarter of the diameter in order that the stress at any part of the section may not change signs, or, in other words, the load-line in cutting any

section must not be outside a circle which is concentric with that forming the outer boundary and whose diameter is $= \frac{d}{4}$.

In the case of a rectangle of height h and width b,

$$k^2 = \frac{h^2}{12}$$

and $$y = \frac{h}{2}$$

so that $$a = \frac{k^2}{y} = \frac{h}{6}$$

This means that the load-line must be kept within the middle third of the depth, in order that the stress may be of the same kind at all parts of the section.

The last result becomes of great importance in such cases as masonry arches and walls, where it is never permissible to put any part of the material under tensile stress. If the load-line gets outside the middle third of the section, the intended compressive stress becomes a tension stress at one edge.

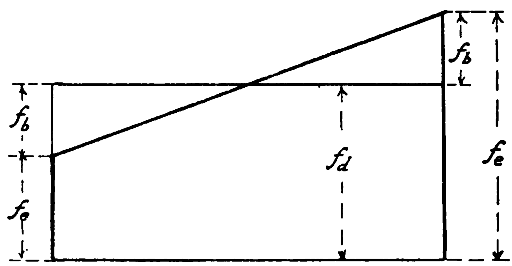

FIG. 60.

In the I section above,

$$k^2 = 16{\cdot}5 \text{ ins.}$$

$$y = 5 \text{ ins.}$$

and $$a = \frac{k^2}{y} = \frac{16{\cdot}5'}{5} = 3{\cdot}3'' \text{ ins.}$$

The depth is 10 ins., so that a can be written as $a = \frac{\text{depth}}{10} \times 3{\cdot}3$, or the load-line must be kept within the middle two-thirds of the depth in the direction of the centre line of the web.

The dotted lines in Figs. 57, 58, 59, indicate the boundaries within which the load-line must act in order to ensure that the stress is all tension or all compression.

The diagram on Fig. 60 shows how the direct stress and the bending stress combine to produce the resultant stress. This is for the case shown on Fig. 64.

Instances of Unequal Stresses due to Eccentric Loading.

The only way to determine the stress occurring in any portion of material under stress is to measure the compression or extension of a given length on the surface of the material. It is well known that so long as the stresses in the material are not carried beyond the elastic limit, they are proportional to the strains or deformations; so that if, when the material is in this condition, the extensions or compressions can be measured on the surface of the material at given points, the deformations so found will be measures of the stresses at these points.

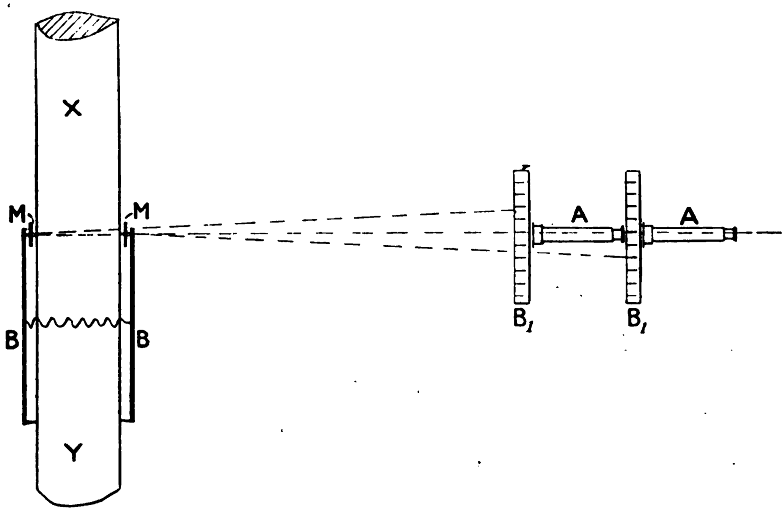

FIG. 61.

Method of Measuring the Strains (Extensions or Compressions).

For the purpose of ascertaining the amount of compression or extension of a structural part or specially prepared specimen, the writer has made use of the Martens mirror apparatus, the principle of whose working is indicated in the sketch on Fig. 61.

Here the material forms part of a bar in tension, XY, and it is desired to measure the extension of the material on the right-hand and left-hand surfaces respectively. Take the right-hand side. Here a light metal bar B is held against the surface of the

material. The lower end of this bar has a sharp edge or point, which penetrates the surface to a very small extent and prevents the measuring bar B from slipping. At the upper end is a groove in which rests one side of a rocking prism, whose other edge rests against the surface of XY. Before loading the prism lies in a horizontal position, but when a tension load is applied the edge of the prism resting against the surface of the material is raised as the bar is extended, while the edge against B remains stationary relatively to the bar; in this way the prism is tilted slightly.

The prism carries a mirror, M, as indicated in the sketch. The observer, on looking through a telescope at A, sees reflected in the mirror a view of a graduated scale B_1, and a hair-line in the diaphragm of the telescope appears to lie across the scale. The more the material of the bar stretches, the further the prism is tilted and a different part of the scale brought into view. The distances and graduations of the scale are so proportioned that the apparent movement of the hair-line across the scale provides a definite measure of the amount of stretch of the material.

It will be seen that in the sketch there are two arrangements of this kind, one on each face of the bar. The apparatus can be used equally well for compression as for tension.

Example 1.—A masonry pier, having the section already given on Fig. 25, carries a load whose line of action passes through a point 1 ft. 6 ins. from the wide edge. To find the maximum load to give a compressive stress which does not exceed 12 tons per sq. ft.:

The moment of inertia, I, has already been found, and = 43·6 foot-units.

Let W = load in tons,

Area of pier = (18+6) = 24 sq. ft.

Uniform compressive stress $= \dfrac{W}{24}$ tons per sq. ft. $= 0{\cdot}0417$ W.

Compressive stress at X due to bending moment

$$= f = \frac{My}{I}$$

$$g = 2{\cdot}875', \text{ and } a = 3{\cdot}500 - 2{\cdot}875 = 0{\cdot}625$$

$$M = (W \times 0{\cdot}625) \text{ lb.-ft.}$$

$$y = (1{\cdot}5 + 0{\cdot}625) = 2{\cdot}125 \text{ ft.}$$

Therefore $$f = \frac{W \times 0{\cdot}625 \times 2{\cdot}125}{43{\cdot}6} = 0{\cdot}030\ W$$

$$\therefore\ 12 = 0{\cdot}0305\ W + 0{\cdot}0417\ W$$

$$\therefore\ W = \frac{12}{0{\cdot}0722} = 166 \text{ tons.}$$

Example 2.—On Fig. 62 is shown a sketch which represents a mild steel test-piece when being subjected to a compressive load between the two platens T T of a testing machine.

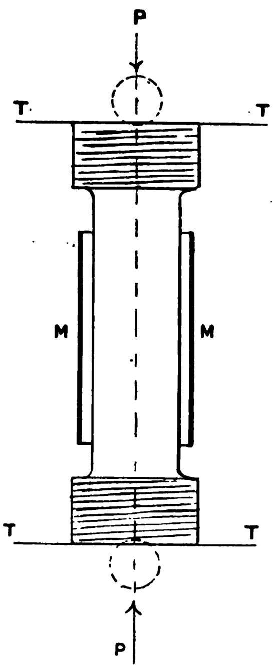

Fig. 62.

The thick lines M M represent the bars B B of the Martens mirror apparatus on Fig. 61. On Fig. 63 are shown curves of loads and compressions for two different conditions of loading.

Those marked (*a*) give the loads and the corresponding compressions when the flat ends of the bar rest against the platens of the machine on the assumption that the line of load will coincide with the axis of the bar and consequently the stress will be the same at all points. That this assumption is not borne out by the observations taken during the experiment is shown by the curves. The eccentricity of the load-line, which gives rise to this inequality of stress, is probably due to the platens of the machine not being quite parallel.

The other pair of curves represents the result of applying the load, not through the flat ends of the bar, but through a pair of steel balls placed in the centres of the ends, and seem to indicate that the load-line is now very near the axis. The two curves almost coincide, showing that the stress is practically uniform.

The figures from which the above curves were plotted were obtained from actual experiment.

Example 3 (Fig. 64).—This is an experiment not unlike the last. Here are two cubical blocks of concrete, one being bedded upon the other through a joint of cement mortar. The Martens mirrors were used as before.

First the load was placed at P P in the centre, and the curves (*a*) (*a*) (Fig. 65) obtained. These almost coincide, indicating uniformity of stress.

The load was then allowed to act at P_1 P_1, just on the edge of the middle third of the section. It has been shown that with

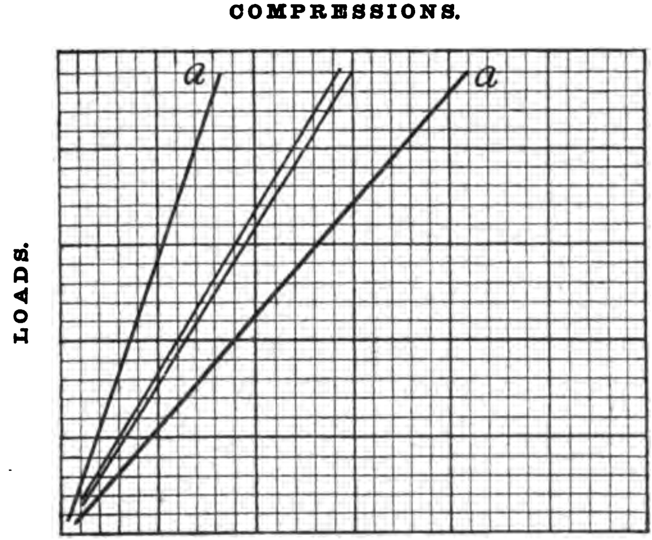

Fig. 63.

this arrangement of loading the compressive stress is a maximum at the right-hand edge and zero on the left-hand edge. This is practically the case in the present experiment, as is shown by the curves *b* and *b*. That the result is not quite what it should be is caused by the difficulty in applying the load precisely at the edge of the middle third.

Example 4.—The two curves on Fig. 66 were obtained from an experiment similar to the last. The sample of material was in this case a brickwork pier 18 ins. square and 36 ins. high (see Fig. 97). The load was applied through a very carefully prepared bed

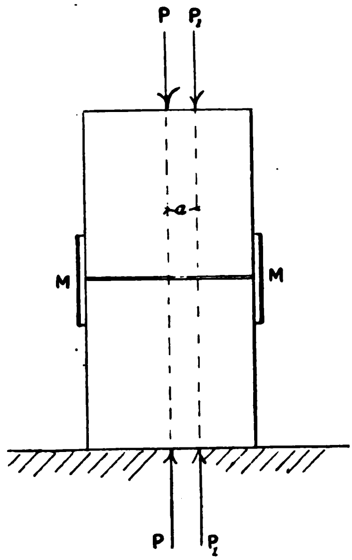

FIG. 64.

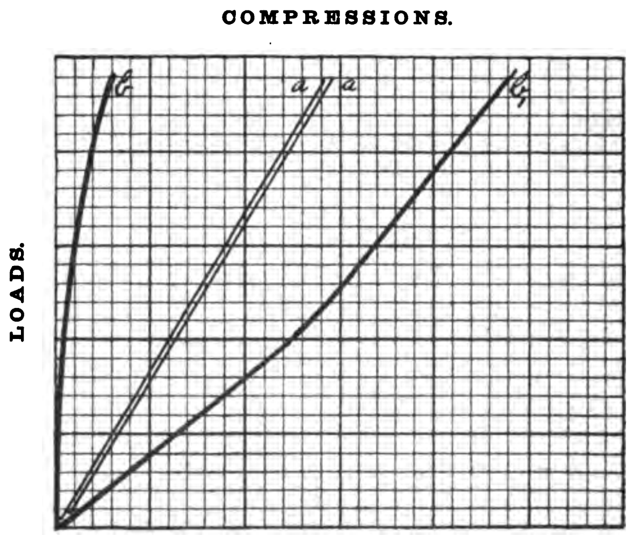

FIG. 65.

of neat cement mortar by first setting the pillar on the lower platen, then putting a layer of soft cement on the top, bringing down the top platen with a little pressure, and leaving it so until the mortar had set. Between the mortar and the platen was a thin

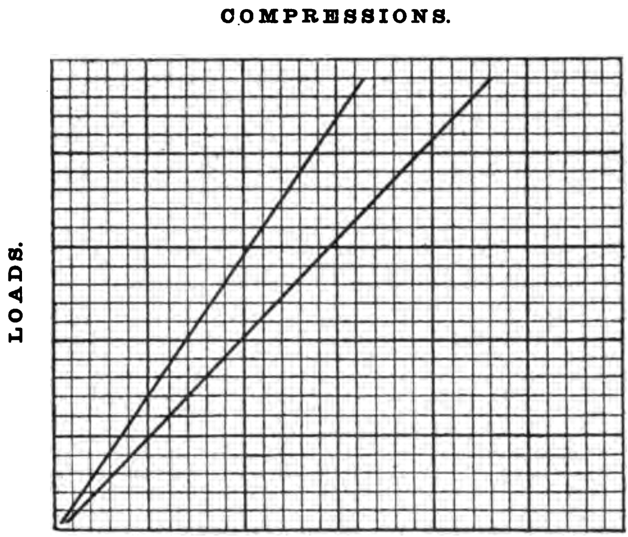

FIG. 66.

layer of cardboard. After this process it would naturally be thought that the stress would be perfectly uniform. The curves indicate that this is far from being the case, and shows how difficult it is to obtain uniformity under these conditions.

CHAPTER VIII

PILLARS, STRUTS, OR COLUMNS

IN the case of a short prism (Fig. 67*a*), whose length is l, smallest diameter d, and area of cross-section A, the crushing load which causes failure is

$$P = fA \quad \text{or} \quad \frac{P}{A} = f$$

where f is the crushing stress for the material in question. For this to hold good, the ratio $\frac{l}{d}$ must not be more than about 3 to 1. Where the ratio becomes much larger than this, P is less than fA, or $\frac{P}{A}$ is less than f.

In this case the structural part is spoken of as a *pillar, strut,* or *long column.*

In a short column failure takes place by simple crushing, and in a long column partly by crushing and partly by bending. The greater the ratio $\frac{l}{d}$ the more is the strength dependent on the effect of the length, until a point is reached, where $\frac{l}{k}$ is greater than about 150, at which the failure is practically due to bending alone and simple crushing is negligible. k = radius of gyration.

For the *first* of the three cases $\frac{P}{A} = f$.

For the *third* case the following formula, due to Euler, is to be

used. *Euler's Formula* for very long columns (Fig. 67*c*), whose ends are perfectly free, is

$$\frac{P}{A} = \pi^2 E \frac{k^2}{l^2}$$

where P, l, and A have the same meanings as before,
E is the direct elastic modulus,
and k is the least radius of gyration of the cross-section.

For the *second* case (Fig. 67*b*), which is the one most commonly

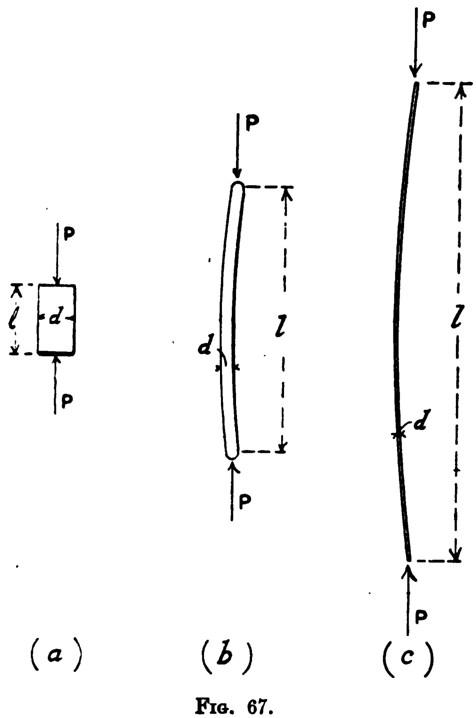

Fig. 67.

met with in practice, various formulas have been devised. It is apparent that the longer the column relatively to its diameter, the more easily will it collapse under a given load, so that a formula must be used where, instead of having $\frac{P}{A}=f$, $\frac{P}{A}$ must be less than f, the diminution becoming greater as l increases. The

formulas in use, of which the three best known are given below, are all more or less empirical.

Gordon's Formula was based by Professor Gordon on the results of Hodgkinson's long series of experiments. It is

$$\frac{P}{A} = \frac{f}{1 + c\dfrac{l^2}{d^2}}$$

Here it will be seen that f is diminished by increasing its denominator, which is made to depend on the square of the ratio $\frac{l}{d}$ and upon an empirical constant c. This constant varies with both the form of the section and the material, with the result that a large number of constants must be known to fit all cases that may arise. The constants originally given for use in this formula hardly apply to the materials and forms of section now in use, and it is not necessary to give them here.

Rankine's Modification of Gordon's Formula is similar in general form, being

$$\frac{P}{A} = \frac{f}{1 + a\dfrac{l^2}{k^2}}$$

In this case the empirical constant a varies only with the material, the difference in shape being accounted for by the change in the radius of gyration k. This is more satisfactory than Gordon's, as fewer constants are required

The following constants are given by several authorities for use in Rankine's formula:—

Material.	f lbs. per sq. in.	a
Wrought iron .	36,000	$\frac{1}{9000}$
Mild steel . .	48,000	$\frac{1}{7500}$
Hard steel. .	70,000	$\frac{1}{5000}$
Cast iron . .	80,000	$\frac{1}{1600}$

In this formula, f is somewhat higher than the *Yield Point Stress* in compression for the first three, which are ductile

materials, and is the *Crushing Stress* for cast iron. As the load on a column increases, there is a small deflection to one side or the other, causing a maximum compressive stress on the concave side. When this stress passes the yield stress of the material, the deflection increases rapidly, and buckling follows.

The constants a, given here, all refer to pillars in which the ends are free to turn, either by being hinged or having rounded ends. The length l is the distance between the centres about which the ends can turn.

Where the column is rigidly fixed at both ends, the proper constant is $\frac{a}{4}$; where it is fixed at one end and free at the other, the constant is $\frac{4}{9}a$.

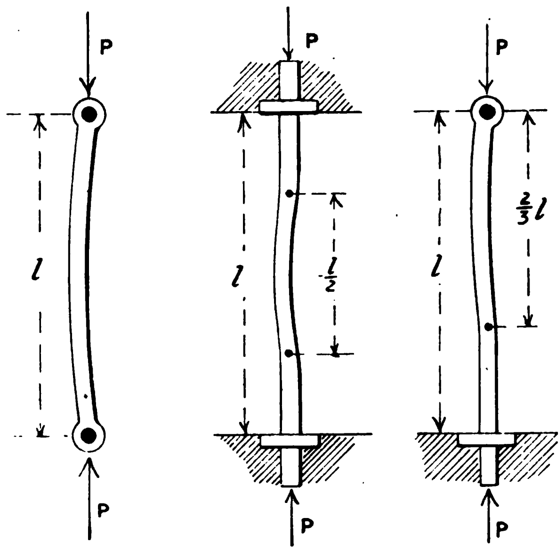

FIG. 68.

Fig. 68 shows the effect of fixing one or both ends. Whatever method of fixing is employed, the effect must be to keep the axis of the unloaded column at the fixed end in the line of action of the load.

A more recent and more workable formula is that of Professor Johnson of Wisconsin, who utilised the results of a large number of destruction tests of columns made by Considère and Tetmajer.

On Fig. 69 are plotted three curves showing the relation between the buckling stress $\frac{P}{A}$ for columns having different values of $\frac{l}{k}$. One shows the values of $\frac{P}{A}$ as given by Euler's formula, which for columns of ordinary lengths is seen to be quite inapplicable. The second is that of Rankine, which gives much more reasonable results. Johnson plotted the results of Considère's and Tetmajer's tests in a similar manner, and found that the parts of the curves between the line of zero $\frac{l}{k}$ and the point of contact with the Euler curve are approximately parabolas whose equations are of the form

$$\frac{P}{A} = f - B\frac{l^2}{k^2} \text{ where } B = \frac{f}{4\pi^2E}$$

Here f is diminished as l increases, not by an increasing divisor, but by taking away a larger quantity.

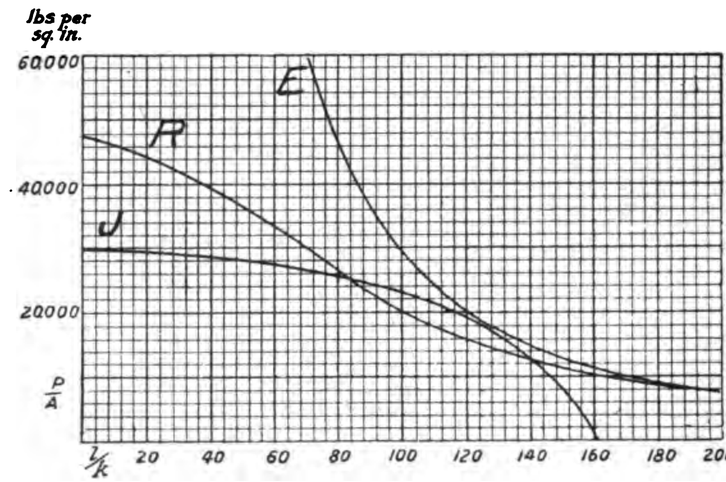

Fig. 69.

The symbol f denotes the compression yield point for the ductile materials and the ultimate crushing stress for brittle materials.

$\frac{P}{A}$ then denotes the stress on the cross-section of the column which will cause it to fail by buckling or fracturing, according to the material. To get the safe stress, $\frac{P}{A}$ must be divided by a factor of safety.

It must also be noted that k is the *least* radius of gyration of the section.

The values given by Johnson for the constants B and f are:—

Table of Values of B and f in Formula $\frac{P}{A} = f - B\frac{l^2}{k^2}$

Material.	Condition of Ends.	$\frac{l}{k}$ not greater than	f lbs. per sq. in.	B.
Wrought Iron	Pin (hinged)	170	34,000	0·67
	Flat	210	34,000	0·43
Mild Steel	Pin (hinged)	150	42,000	0·97
	Flat	190	42,000	0·62
Cast Iron	Rounded	70	60,000	6·25
	Flat	120	60,000	2·25

It is necessary to point out that the values of f which are given above are those chosen by Johnson, as average values arising from his own experience. Where accuracy is desired, it will be necessary to find the value of f directly from compression tests of samples of the material in question, and calculate B accordingly.

The following example will serve to illustrate the use of Johnson's formula:—

Steel pillar of the section shown on Fig. 70, loaded through rollers whose centres are 49¼ inches apart.

Depth = 6 ins.
Width of flange = 3 ins.
Thickness of flange = 0·45 in.
" of web = 0·39 in.

From the dimensions it is found that

$$A = 4{\cdot}66 \text{ sq. ins.}$$
$$I = 2{\cdot}05.$$
$$k^2 = \frac{I}{A} = 0{\cdot}440.$$
$$l = 49{\cdot}25 \text{ ins.}$$
$$f = 40{,}000 \text{ lbs. per sq. in. by experiment.}$$

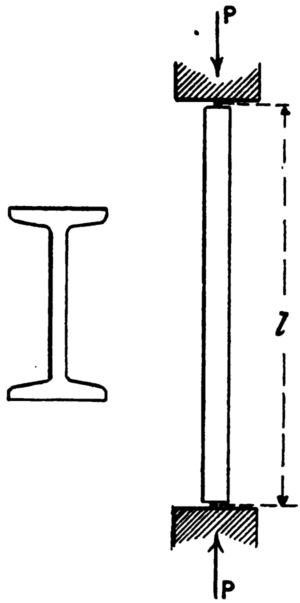

Fig. 70.

Then,

$$P = A\left(f - B\frac{l^2}{k^2}\right)$$

and

$$\left[B = \frac{f^2}{4\pi^2 E} = 1{\cdot}33\right]$$

$$P = 4{\cdot}66\left(40{,}000 - 1{\cdot}33\frac{(49{\cdot}25)^2}{0{\cdot}440}\right)$$
$$= 68 \text{ tons nearly.}$$

The actual collapsing load of this was found by experiment to be 69 tons.

CHAPTER IX

TORSION AND SPRINGS

By torsion is meant the effect produced by the action of a pair of equal and opposite couples at the ends of a prism or shaft acting in planes at right angles to its axis. The kind of stress caused by torsion most commonly occurs in rotating shafts used for transmitting power.

Elastic Circular Shaft.—In Fig. 71 is shown a view of a short length, l, of a solid circular shaft, whose ends consist of two

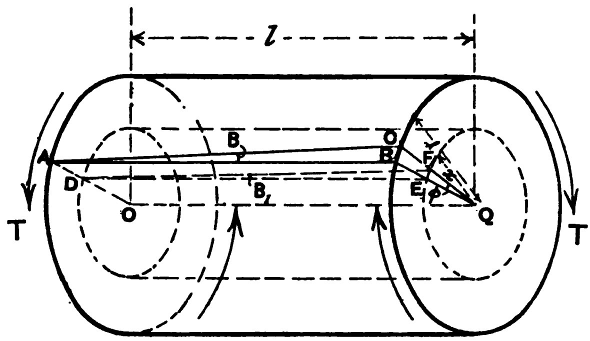

Fig. 71.

circles having centres O and Q at the ends of the axis OQ. These circular planes are perpendicular to the axis.

This forms a short length taken out of a shaft such as that shown on Fig. 72.

Suppose a line AB parallel to the axis to be drawn on the surface of the shaft before any force is applied to it.

The pair of equal and opposite couples or twisting moments, T T, are now allowed to act on the shaft. The effect will be to rotate the ends relatively to one another, in the directions

indicated by the arrows. If A be taken as stationary, the effect of the twisting will be to move the point B to C, and the line AB into a new position AC, making an angle β with AB. This line AC will form part of a spiral.

The movement of B to C will mean that the radius $QB=r$ takes up a new position QC after turning through an angle $BQC=\phi$. For the purposes of the following proof the material is assumed to be elastic. This being the case, it is reasonable to assume that the radius QB, which is a straight line, remains

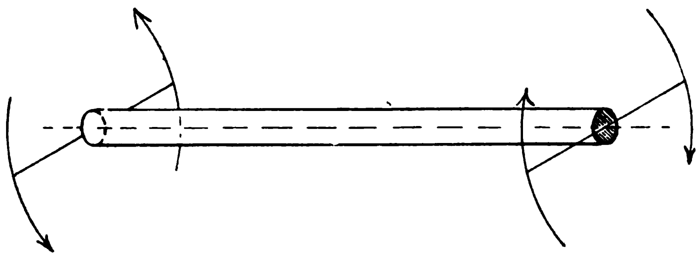

FIG. 72.

straight when twisting takes place. If E is any point in this radius intermediate between B and Q, where $QE=x$, E will move to F when B moves to C, and

$$\frac{EF}{BC} = \frac{x}{r}$$

From this it follows that a line DE, which is parallel to the axis before the twist is applied, will assume a new position DF, the angle EDF being called β_1.

Now, $$\tan\beta = \frac{BC}{AB}, \text{ and } \tan\beta_1 = \frac{EF}{DE} = \frac{EF}{AB}$$

or $$\frac{\tan\beta_1}{\tan\beta} = \frac{EF}{BC} = \frac{x}{r}$$

For stresses within the elastic limit, the angles of distortion are relatively small, and the above may be written

$$\frac{\beta_1}{\beta} = \frac{x}{r}$$

This angular distortion is due to the shear stress produced by the twisting moment and acting circumferentially about Q. It

has previously been shown that the angle of distortion is proportional to the shear stress producing it. Or, if f_s is the shear stress at BC, on the surface of the shaft, and s is the shear stress at any other surface EF, then

$$\frac{s}{f_s} = \frac{\beta_1}{\beta} = \frac{x}{r}$$

or, in other words, the shear stress in a circular shaft subjected to torsion has a maximum value at the surface of the shaft, and at any other radius x is proportional to that radius.

Next, to find the torsional resisting moment. On Fig. 73 the circle represents the section of the shaft having a diameter d and radius r. Let this section be divided up into a number of elemental rings of radius $= x$, and width $= \delta x$. The area of one such ring $= 2\pi x \delta x$. The circumferential force on the ring $= s\,(2\pi x \delta x)$

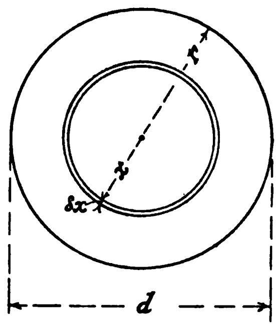

FIG. 73.

$$= \left(\frac{f_s}{r}x\right)\left(2\pi x\delta x\right)$$

as $$s = f_s\frac{x}{r}$$

Also, the moment of this force about the centre is

$$= x\left(\frac{f_s}{r}x\right)\left(2\pi x\delta x\right)$$

$$= \frac{2\pi}{r}f_s x^3\delta x$$

This is the torsional resisting moment of one of the elemental rings, and the total resisting moment will be

$$= \frac{2\pi f_s}{r}\int_0^r x^3 dx = \frac{\pi}{2}f_s r^3$$

This is equal and opposite to the external twisting moment T, or

$$T = \frac{\pi}{2}f_s r^3 = \frac{\pi}{16}f_s d^3, \text{ where } d = 2r$$

For a hollow shaft (Fig. 74) having inner and outer diameters respectively $=d_2$ and d_1, and radii r_2 and r_1,

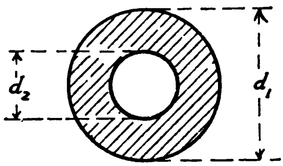

FIG. 74.

$$T = \frac{2\pi f_s}{r}\int_{r_2}^{r_1} x^3 dx$$

$$= \frac{\pi}{2} f_s \frac{r_1^4 - r_2^4}{r_1}$$

$$= \frac{\pi}{16} f_s \frac{d_1^4 - d_2^4}{d_1}$$

The expressions

$$\frac{\pi}{2} r^4,\ \frac{\pi}{32} d^4,\ \frac{\pi}{2}\left(r_1^4 - r_2^4\right),\ \text{and}\ \frac{\pi}{32}\left(d_1^4 - d_2^4\right)$$

are the polar moments of inertia, I_p, where

$$\frac{T}{I_p} = \frac{f_s}{r}$$

Angle of twist of a Shaft.—To find the angle ϕ through which the circular end at B is rotated relatively to that at A,

$$\frac{BC}{r} = \phi\ \text{(radians), or}\ BC = \phi r$$

Also for small angles,

$$\frac{BC}{AB} = \frac{BC}{l} = \beta\ \text{(radians), or}\ BC = \beta l$$

So that
$$BC = \phi r = \beta l\ \text{ or }\ \phi = \beta\frac{l}{r}$$

Again, for the angular distortion $\beta = \frac{f_s}{G}$, where G is the shear modulus; so that the above may be written

$$\phi = \frac{fs}{G}\frac{l}{r} = \frac{2T}{\pi}\frac{l}{Gr^4}\ \text{(radians), or}\ \frac{32T\,l}{\pi G d^4}$$

because f_s has been shown to $= \frac{2T}{\pi r^3}$

For hollow shafts

$$\phi = \frac{2Tl}{\pi G\left(r_1^4 - r_2^4\right)}\ \text{(radians)}$$

Expressed in degrees, these angles are respectively

$$\phi^\circ = \frac{2T}{\pi G}\frac{l}{r^4}\frac{180}{\pi} = \frac{360\,T\,l}{\pi^2 G\,r^4}$$

For hollow shafts

$$\phi^\circ = \frac{360\,T\,l}{\pi^2 G\,(r_1^4 - r_2^4)}$$

Using diameters instead of radii

$$\phi^\circ = \frac{5760\,T\,l}{\pi^2 G\,d^4}$$

and for hollow shafts,

$$\phi = \frac{5760\,T\,l}{\pi^2 G\,(d_1^4 - d_2^4)}$$

It is the usual practice not to allow ϕ° to exceed 1° in a length $l = 20d$.

Horse-power transmitted.—If T is expressed in lb.-inches, then $\frac{T}{12}$ will be the twisting moment expressed in lb.-feet. And the work done by T in one minute will be $\frac{TN2\pi}{12}$ where N is the number of revolutions made by the shaft in one minute. The horse-power transmitted will therefore be

$$\text{H.P.} = \frac{TN2\pi}{12 \times 33{,}000} = \frac{\pi}{16}\,\frac{f_s d^3 N 2\pi}{\times 12 \times 33{,}000}$$

from which approximately,

$$\text{H.P.} = \frac{f_s d^3 N}{322{,}000}$$

From this the diameter d of a shaft to transmit a given H.P. at speed of revolution, N, can be found, as

$$d = \sqrt[3]{\frac{322{,}000\ \text{H.P.}}{f_s N}} = 67{\cdot}8\sqrt{\frac{\text{H.P.}}{f_s N}}$$

The usual values given to f_s in practice are:

For wrought iron shafts $f_s = 5000$ lbs. per sq. in.
„ mild steel shafts $f_s = 6000$ lbs. per sq. in.
„ medium-carbon steel shafts $f_s = 8000$ lbs. per sq. in.

Torsional Strength of Shafts in the Plastic State.

When the twisting moment on a shaft is increased to such a point that the material is being stressed some way beyond the limit of elasticity, experiment seems to show that the shear stress s at any radius x is no longer proportional to x, but has become much more nearly constant and equal to the stress f_s at the surface. The author has found that in ductile materials like soft steel and the best wrought iron s is practically $= f_s$ when the point of fracture is reached. Therefore, taking s as being equal to f_s, the former equations become:

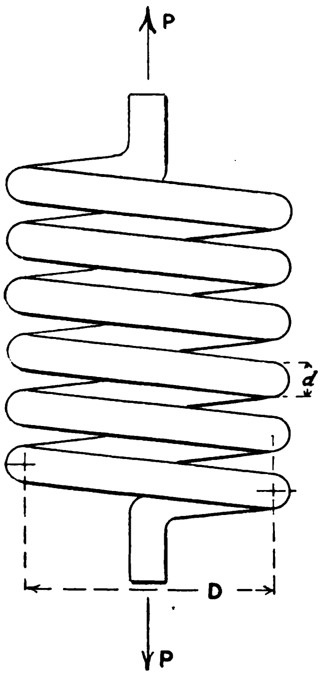

Fig. 75.

For solid shafts,

$$T = 2\pi f_s \int_0^r x^2 dx = \tfrac{2}{3}\pi f_s r^3 = \frac{\pi}{12} f_s d^3$$

And, for hollow shafts,

$$T = \tfrac{2}{3}\pi f_s (r_1^3 - r_2^3) = \frac{\pi}{12} f_s (d_1^3 - d_2^3)$$

For shafts under working conditions the elastic limit is never reached, and the former set of formulæ are to be used for providing a relation between the working torsional moment, the safe shear stress, and the dimensions of the shaft.

Where it is desired to find the twisting moment necessary to fracture the shaft, the latter set of formulæ are to be used, where f_s is the ultimate shearing strength of the material.

Loads and Deformations of Springs.

Helical springs of round steel.—In Fig. 75 is shown a spring of this type intended to be loaded in tension. The following applies equally well to a compression spring.

Here, let

P = the load on the spring acting along its axis.
D = the mean diameter of the coil.
d = the diameter of the wire.
n = number of complete coils.
G = shear modulus of the material.

Then the length of wire in the spring,

$l = \pi D n$

The effect of the load is to put upon the wire, throughout its length, a torsional moment whose value is

$$T = \frac{PD}{2}; = PR \text{ where R is the radius of the coil,}$$

and this results in a twist of the wire, as shown in Fig. 76.

Consider one coil. Throughout the length l of the coil there is the twisting moment T, due to a force P acting at the end of an arm $\frac{D}{2}$. The twisting of the wire will allow the point A to move relatively to B an amount δ where

$$\frac{\delta}{\frac{D}{2}} = \phi$$

ϕ being the angle of twist for the length in question, or.

$$\delta = \phi \frac{D}{2}$$

But it has previously been shown that

$$\phi = \frac{32Tl}{\pi G d^4}$$

so that $$\delta = \frac{8PD^3}{Gd^4}$$

for one coil. The total extension for a spring of n coils will then be

$$\triangle = \frac{8PnD^3}{Gd^4}$$

The corresponding formula for springs of square metal is:

$$\triangle = \frac{5{\cdot}88PnD^3}{Gd^4}$$

It is to be noted that the above are only true so long as the strains are relatively small, so that δ is sensibly $= \frac{\phi D}{2}$, and not $= \tan\phi \frac{D}{2}$

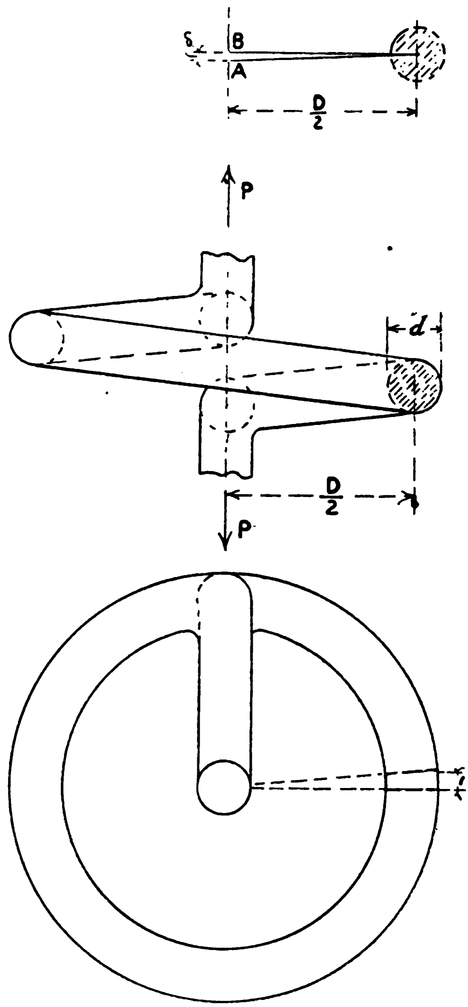

Fig. 76.

For spring steel the value of G varies from 12,000,000 lbs. per sq. in. to 14,000,000 lbs. per sq. in.

Example 1.—A solid steel shaft is to transmit 600 H.P. at a speed of 58 revolutions per minute. Find the diameter of the shaft. The stress in the metal must not exceed 8000 lbs. per sq. in.

$$\text{H.P.} = \frac{2\pi N \frac{\pi}{16} f_s d^3}{12 \times 33{,}000}$$

and

$$d = \sqrt[3]{\frac{\text{H.P.} \times 33{,}000 \times 12}{2\pi N \frac{\pi}{16} f_s}}$$

$$= \sqrt[3]{\frac{600 \times 33{,}000 \times 12 \times 16}{2 \times \pi^2 \times 58 \times 8000}}$$

$$= 7{\cdot}46 \text{ ins.}$$

Example 2.—A shaft 3 ins. in diameter, running at 250 revolutions per minute, transmits 50 H.P. Find the maximum stress.

Using the same symbols as above,

$$\text{H.P.} = \frac{2\pi N \frac{\pi}{16} f_s d^3}{12 \times 33{,}000}$$

and

$$f_s = \frac{\text{H.P.} \times 12 \times 33{,}000}{2\pi N \frac{\pi}{16} d^3}$$

$$= \frac{50 \times 12 \times 33000 \times 16}{2 \times \pi^2 \times 250 \times (3)^3}$$

$\therefore$ The maximum stress = 2378 lbs. per sq. in.

Example 3.—A spring of steel of circular section containing five coils is subjected to compressive loads in a testing machine. Loads are applied and readings taken of the lengths. A piece of the steel is then subjected to tension loads. Loads are applied and readings taken on a length of 15 cm. The outside diameter of the spring is 4·8 ins. and the diameter of the steel 0·96 in. Find the shear modulus and the modulus of direct elasticity.

Let G = shear modulus in lbs. per sq. in.
E = modulus of elasticity in lbs. per sq. in.
P = load on spring (compressive) in lbs.
D = mean diameter of spring in inches.
d = diameter of the steel in inches.
$\triangle$ = the compression on the spring in inches.
n = the number of coils in the spring.

Mean diameter of spring,

$$D = 4{\cdot}8 - 0{\cdot}96$$
$$= 3{\cdot}84 \text{ ins.}$$

TABLE OF READINGS TAKEN WHEN THE SPRING WAS COMPRESSED.

Loads, tons . . .	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$
Length of 5 coils, ins.	7·39	7·24	7·09	6·95	6·81	6·66	6·52
Compressions . .	—	0·15	0·15	0·14	0·14	0·15	0·14

Loads, tons . . .	2	$2\frac{1}{4}$	$2\frac{1}{2}$	$2\frac{3}{4}$	3	$3\frac{1}{4}$
Length of 5 coils, ins.	6·38	6·25	6·11	5·98	5·84	5·67
Compressions . .	0·14	0·13	0·14	0·13	0·14	0·17

The average compression (from $\frac{1}{4}$ to 3 tons) for $\frac{1}{4}$ ton

$$= \frac{1{\cdot}55}{11} = 0{\cdot}1409 \text{ in.}$$

The shear modulus

$$G = \frac{8\,\mathrm{P}n\mathrm{D}^3}{\triangle\, d^4}$$
$$= \frac{8 \times 560 \times 5 \times (3{\cdot}84)^3}{0{\cdot}1409 \times (0{\cdot}96)^4}$$
$$= 10{,}590{,}000 \text{ lbs. per sq. in.}$$

In the tension test on a piece of the same steel, the readings were taken on 15 cm. = 5·905 ins.

Let E = the elastic modulus in lbs. per sq. in.
l = the length on which readings were taken in inches.
f = the stress in the metal in lbs. per sq. in.
x = the elongation in inches.
W = the load applied to produce x, in lbs.
A = the cross-sectional area of the metal in sq. ins.

TABLE OF EXTENSIONS.

Load in Tons	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2
Extensions, $\frac{1}{1000}$th in.	0	0·14	0·29	0·45	0·605	0·76	0·915	1·065
Differences	—	0·14	0·15	0·16	0·155	0·155	0·155	0·15

Load in tons	$2\frac{1}{4}$	$2\frac{1}{2}$	$2\frac{3}{4}$	3	$3\frac{1}{4}$	$3\frac{1}{2}$	$3\frac{3}{4}$	4
Extensions, $\frac{1}{1000}$th in.	1·22	1·38	1·525	1·68	1·825	1·985	2·14	2·295
Differences	0.155	0·16	0·145	0·155	0·145	0·16	0·155	0·155

The average extension (from $\frac{1}{4}$ to 4 tons) for $\frac{1}{4}$ ton

$$= \frac{2·295}{15} = 0·153 \ \frac{1}{1000}\text{ths of an in.,}$$

i.e., $x = 0·000153$ in.

The stress per sq. in. of sectional area of the metal

$$f = \frac{W}{A}$$

The modulus of elasticity

$$\begin{aligned} E &= \frac{fl}{x} \\ &= \frac{Wl}{Ax} \\ &= \frac{560 \times 5·905}{0·7854 \times (0·96)^2 \times 0·000153} \\ &= 29{,}860{,}000 \text{ lbs. per sq. in.} \end{aligned}$$

The ratio of

The modulus of elasticity : The shear modulus
is as:—29,860,000 : 10,590,000
i.e., 2·819 : 1

CHAPTER X

TORSION COMBINED WITH BENDING

IT often occurs in practice that a shaft has not only to withstand a torsional moment T, but is so supported and loaded that there is a bending moment acting at the same time. The most usual instances of this kind occur in shafts which run in bearings some distance apart and carry heavy wheels or pulleys between the bearings, and in the crank-shafts of engines.

On Fig. 77 is shown a typical case. A force acts at right

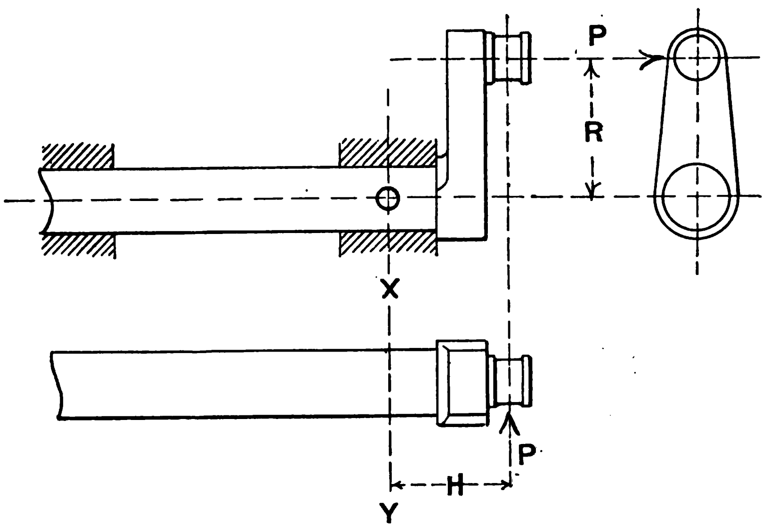

FIG. 77.

angles to the centre line of a crank of radius R, giving rise to a twisting moment

$$T = PR$$

In plan the line of action of P is distant H from the centre of the

nearer bearing, so that on the section of the shaft at XY there is, in addition to T, a bending moment

$$M = PH$$

On Fig. 78 is shown an enlarged view of the portion of the shaft near O, the force P being supposed normal to the paper. In the centre of the shaft, at the point marked by the circle and on the section made by XY, there will be a maximum direct stress due to M. If P acts *towards* the paper, this will be a tensile stress; if *from* the paper, the material will be in compression at this point. At the same point there will also be a shear stress caused by the twisting moment T. Call the direct stress f and the shear stress f_s. There will be a shear stress along YX, and it has previously been shown that there must be a shear stress of equal intensity on planes at right angles to YX.

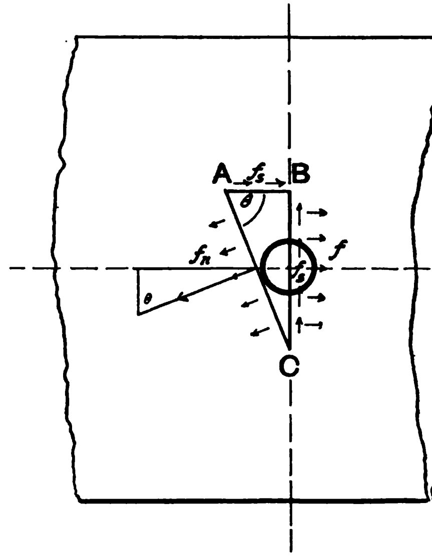

Fig. 78.

Next, consider an elemental triangular solid ABC, of uniform thickness, at the surface of the shaft, and which is kept in equilibrium by the stresses acting on its faces, and which is of such proportions that the resultant stress in its diagonal AC is wholly normal. Call the angle CAB $= \theta$.

On AB is the shear stress f_s, on BC is the direct stress f and also the shear stress f_s, and on AC is the resulting normal stress f_n.

The total normal stress on AC $= f_n$AC. Resolving this horizontally and vertically:

$$\text{(Horizontal component)} = f_n\text{AC} \sin\theta = f\,\text{BC} + f_s\text{AB}$$

$$\text{(Vertical component)} \quad = f_n\text{AC} \cos\theta = f_s\text{BC}$$

From which

$$f_n - f = \frac{f_s AB}{BC} = f_s \cot\theta \qquad (a)$$

and

$$f_n = f_s \frac{BC}{AB} = f_s \tan\theta \qquad (b)$$

Subtracting (*b*) from (*a*),

$$-f = f_s(\cot\theta - \tan\theta)$$
$$= 2f_s \cot 2\theta$$

or,

$$\cot 2\theta = -\frac{f}{2f_s}$$

and

$$\tan 2\theta = -\frac{2f_s}{f} \qquad (1)$$

This gives two values of 2θ differing by 180°, and two values of θ differing by 90°.

Now multiply the above two equations (*a*) and (*b*) together:

$$(f_n - f)(f_n) = f_s^2 \cot\theta \tan\theta = f_s^2$$

This last equation will be made use of for combining the torsion and bending moments.

Here $\quad f = \frac{Mr}{I} = \frac{4M}{\pi r^3}$

and $\quad f_s = \frac{2T}{\pi r^3}$

Now $\quad f_n^2 - ff_n = f_s^2$

This is a quadratic whose solution gives

$$f_n = \frac{f}{2} \pm \sqrt{\frac{f^2}{4} + f_s^2}$$
$$= \frac{2M}{\pi r^3} \pm \sqrt{\frac{4M^2}{(\pi r^3)^2} + \frac{4T^2}{(\pi r^3)^2}}$$
$$\frac{\pi r^3}{2} f_n = M \pm \sqrt{M^2 + T^2}$$

Putting $\frac{\pi r^3}{2} f_n = T_E$ where T_E is a twisting moment which, acting alone on the shaft in question, would produce a stress of magnitude f_n in the material, and is called the *equivalent twisting*

moment, the above may be written, taking the *plus* value as being the greater,

$$T_E = M + \sqrt{M^2 + T^2}$$

or, if M_E is the *equivalent bending moment,*

$$M_E = \frac{1}{2}\left(\frac{\pi r^3}{2} f_n\right)$$

that is,

$$M_E = \frac{1}{2}M + \frac{1}{2}\sqrt{M^2 + T^2}$$

The stress f_n, brought about by the torsion and bending moments acting simultaneously, may be taken as being a tensile stress on AC, a compressive stress on HG at right angles to AC, or a shear stress on planes making angles of 45° with these. See Fig. 79.

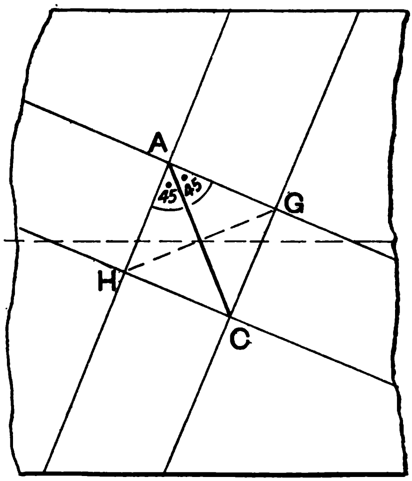

FIG. 79.

From equation (1), knowing f_s and f, it is possible to calculate the angle which the plane, whose stress is wholly normal, makes with the axis.

Where there is no bending, and $f = o$,

$$\text{Tan } 2\theta = \text{(infinity)}$$
$$2\theta = 90°$$

and $\theta = 45°$

That this is so is very forcibly illustrated in the case of a cast iron shaft broken under pure torsion, when it is seen that the surface of failure takes a spiral form, the helix making an angle of 45° with the axis. Such a fracture is shown on Fig. 80 (a).

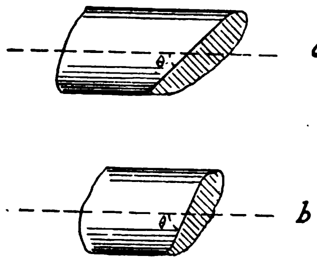

FIG. 80.

Where there is bending as well as torsion, the angle θ is greater than as indicated in Fig. 80 (b).

The formula

$$M_E = \tfrac{1}{2}M + \tfrac{1}{2}\sqrt{M^2 + T^2}$$

is the one given by Rankine, but, according to the more complete theory of elasticity, when Poisson's Ratio is taken into account, this is not strictly true. Grashof gives the more correct value as

$$M_E = \tfrac{3}{8}M + \tfrac{5}{8}\sqrt{M^2 + T^2}$$

Example.—A steel shaft is supported in two bearings 40 ins. from centre to centre, and midway between these carries a pulley weighing 10 tons. 410 H.P. are transmitted through the shaft at 98 revolutions per minute. Find the diameter when the stress in the material does not exceed $3\frac{1}{2}$ tons per sq. in.

Here $$M = \frac{10 \times 20}{2} = 100 \text{ in.-tons,}$$

and $$T = \frac{(\text{H.P.})\ 12\ 33{,}000}{2\pi N 2240} = 118 \text{ in.-tons.}$$

Then
$$\begin{aligned} T_E &= M + \sqrt{M^2 + T^2} \\ &= 100 + \sqrt{(100)^2 + (118)^2} \\ &= 255 \text{ in.-tons.} \end{aligned}$$

And, finally, the diameter

$$\begin{aligned} d &= \sqrt[3]{\frac{T_E 16}{\pi f_s}} \\ &= \sqrt[3]{\frac{255 \times 16}{3{\cdot}14 \times 3{\cdot}5}} \\ &= 7{\cdot}2 \text{ ins.} \end{aligned}$$

CHAPTER XI

STRENGTH OF CYLINDERS

Thin Cylinders.—In a cylinder of length l (see Fig. 81), whose thickness, t, is small relatively to its internal diameter, d, and which is subjected to an internal pressure of intensity, p, the total force tending to lift the upper half from the lower half is pdl, or $p2rl$ where $r = \frac{d}{2}$ = the internal radius.

To this separation is opposed the resisting stress in the material. The area on which this stress f acts is tl on each side, or, altogether, $2tl$, so that the total force opposing pdl is $f2tl$. Therefore,

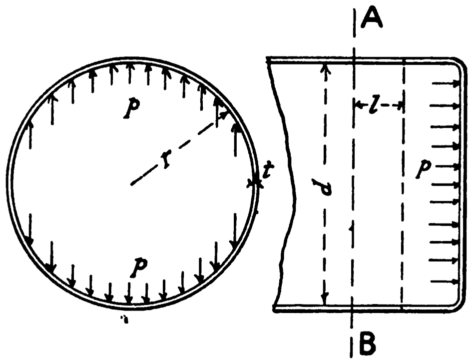

FIG. 81.

$$pdl = 2ftl, \text{ or } pd = 2ft, \text{ or } pr = ft.$$

If the ends of the cylinder are closed, as in the case of a boiler, there is a stress f_e induced on the annular section made by a plane AB at right angles to the axis of the cylinder. Now the total pressure on the end is $p\frac{\pi}{4}d^2$.

This is resisted by f_e acting on a ring whose area is very nearly πdt. So that,

$$p\frac{\pi}{4}d^2 = f_e \pi dt$$

$$pd = 4f_e t$$

or

$$f_e = \frac{pd}{4t} = \frac{pr}{2t}$$

But, from the last equation, the stress on a longitudinal section is $\frac{pr}{t}$, and, therefore,

$$\frac{\text{Stress on a transverse section}}{\text{Stress on a longitudinal section}} = \frac{f_e}{f} = \frac{\frac{pr}{2t}}{\frac{pr}{t}} = \frac{1}{2}$$

Thick Cylinders.—In what has just been said about thin cylinders, where the ratio $\frac{d}{t}$ is large, the stress is assumed to be sensibly uniform throughout the thickness.

Where $\frac{d}{t}$ is small and has a value of from about 8 to 3, the cylinder is called *thick*, and must be treated in a different manner.

For a cylinder of this kind, subjected to an internal pressure, the greatest tensile stress is on the inside surface. The pressure upon the thin cylinder lying next the inside is balanced partly by the resistance to stress of its own material and partly by the support it receives from the material which lies immediately outside it. So that the internal pressure upon the second layer is less than that upon the first, and so on, each successive layer from inside to outside being subjected to a smaller tensile stress. The precise nature of the stress variation in the walls of a thick cylinder is given in the following. The section of such a cylinder is shown in Fig. 82.

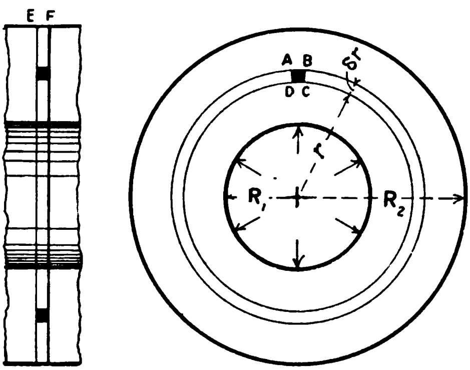

FIG. 82.

In this proof it is assumed that a plane section of the cylinder before pressure is applied remains plane when under stress; that is, there is no distortion of the plane section. The material is supposed to be elastic, and it must also be supposed that the stresses in the material only vary with the distance from the centre—*i.e.*, at the same radius there is the same stress. Take a

slice of the cylinder of thickness EF. Consider the elemental ring of thickness δr. Since the stress is practically uniform throughout this ring, it is only necessary to consider the small portion of it ABCD. This portion is shown more in detail on Fig. 83.

Three stresses act on this piece, namely,

p, radially,
f, circumferentially,
and f_1, longitudinally.

These are taken as having *plus* signs when acting away from the centre of the elemental cube ABCD. The strains, radially, circumferentially, and longitudinally, caused by the three stresses, are:

Due to stress p.	*Due to stress f.*	*Due to stress f_1.*
$a_1 = \frac{-p}{\mu E}$	$a_1 = \frac{-f}{\mu E}$	$a_1 = \frac{f_1}{E}$
$a_2 = \frac{-p}{\mu E}$	$a_2 = \frac{f}{E}$	$a_2 = \frac{-f_1}{\mu E}$
$a_3 = \frac{p}{E}$	$a_3 = \frac{-f}{\mu E}$	$a_3 = \frac{-f_1}{\mu E}$

$\frac{1}{\mu}$ being Poisson's Ratio. Hence total strains are:

$$A_1 = \frac{f_1}{E} - \frac{(p+f)}{\mu E}$$
$$A_2 = \frac{f}{E} - \frac{(p+f_1)}{\mu E}$$
$$A_3 = \frac{p}{E} - \frac{(f+f_1)}{\mu E}$$

Now, as the plane normal section is always plane, the longitudinal strain A_1 will be a constant, but $\frac{f_1}{E}$ is a constant, and often $f_1 = 0$. Hence, $p + f =$ constant. This may be put:

$$p + f = 2X \qquad (1)$$

For radius r, by theory of thin cylinders, $2pr =$ force tending to fracture ring outwardly.

Also, $2(p+\delta p)(r+\delta r)$ = force tending to crush ring inwards and $2f\delta r$ = circumferential force tending to push ring outwards.

These forces balance, hence

$$2(p+\delta p)(r+\delta r)=2pr+2f\delta r, \text{ or } (p+\delta p)(r+\delta r)-pr=f\delta r$$

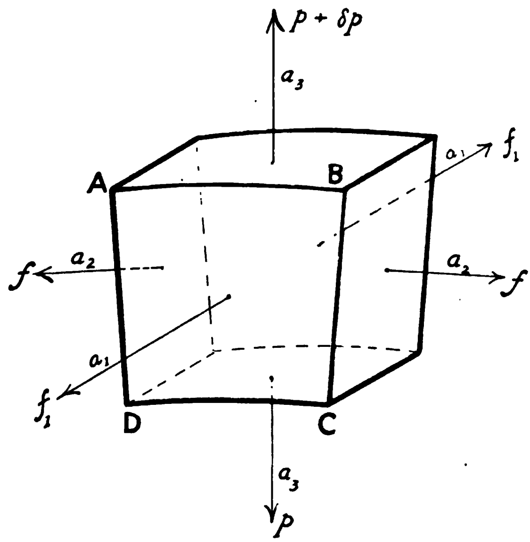

FIG. 83.

Hence, in limit, when δp and δr are indefinitely diminished,

$$d(pr)=fdr$$

Hence

$$p+r\frac{dp}{dr}=f \qquad (2)$$

This gives

$$2p+r\frac{dp}{dr}=p+f=2X,$$

or,

$$2pr+r^2\frac{dp}{dr}=2Xr$$

Hence, integrating, $pr^2=Xr^2+Y$ (Y = constant). Therefore,

$$p=X+\frac{Y}{r^2}$$

and $$f=X-\frac{Y}{r^2} \qquad (3)$$

These are the fundamental equations, and may be applied to the following cases:—

Case of Hydraulic Cylinder.

Let P = internal pressure and P_a = external pressure, from (3),

$$P = X + \frac{Y}{R_1^2}$$

$$P_a = X + \frac{Y}{R_2^2}$$

therefore,

$$Y = \frac{(P - P_a) R_1^2 R_2^2}{R_2^2 - R_1^2} \text{ and } X = \frac{-(PR_1^2 - P_a R_2^2)}{R_2^2 - R_1^2}$$

and

$$f = -\frac{PR_1^2 - P_a R_2^2}{R_2^2 - R_1^2} + \frac{(P - P_a) R_1^2 R_2^2}{(R_2^2 - R_1^2) r^2} \qquad (4)$$

Neglecting external pressure, $P_a = 0$, the maximum stress (at R_1) is given by

$$F = -\frac{P(R_2^2 + R_1^2)}{R_2^2 - R_1^2} \qquad (5)$$

Case of external pressure being great compared with internal pressure.

In this case $P = 0$. Equation (4) then gives:

$$F = f = + \frac{P_a R_2^2 + P_a R_2^2}{R_2^2 - R_1^2} = \frac{2P_a R_2^2}{R_2^2 - R_1^2} \qquad (6)$$

This is evidently a compressive stress.

Example 1.—A steel boiler 8 ft. diameter, with working pressure 150 lbs. per sq. in. and safe stress = 6 tons per sq. in. To find t:

$$\frac{p}{f} = \frac{t}{R}$$

$$\therefore\ t = \frac{Rp}{f}$$

$R = 4$ ft. $= 48$ ins. $\quad f = 6$ tons per sq. in. $\quad p = 150$ lbs. per sq. in.

$$\therefore\ t = \frac{48 \times 150}{6 \times 2240} = 0{\cdot}53 \text{ in.}$$

Example 2.—Hydraulic cylinder, 6 ins. inside radius, internal pressure, $p=1100$ lbs. per sq. in. Safe $F=1$ ton per sq. in. Find thickness:—

$$\frac{P}{F} = \frac{R_2^2 - R_1^2}{R_2^2 + R_1^2} = \frac{1 - \dfrac{R_1^2}{R_2^2}}{1 + \dfrac{R_1^2}{R_2^2}}$$

$$P\left(1 + \frac{R_1^2}{R_2^2}\right) = F\left(1 - \frac{R_1^2}{R_2^2}\right)$$

$$1100 + \frac{1100 \times 6^2}{R_1^2} = 2240 - \frac{2240 \times 6^2}{R_2^2}$$

$$(1100R_2^2 + 1100 \times 36) = (2240R_2^2 - 2240 \times 36)$$

$$1140R_2^2 = 3340 \times 36 = 120100$$

$$R_2^2 = 105{\cdot}3$$

$$R_2 = 10{\cdot}2$$

Thickness of material $= 10{\cdot}2 - 6 = 4{\cdot}2$ ins.

Variation of circumferential and radial stresses in thick cylinders subjected to internal pressure P *alone.*

This is the most usual case in practice. Here $P_a=0$ and $f=$ circumferential stress at any radius r

$$= \frac{-PR_1^2}{r^2} \times \frac{r^2 + R_2^2}{R_2^2 - R_1^2}$$

and $p=$ radial stress at any distance r from the centre,

$$= \frac{PR_1^2}{r^2} \times \frac{R_2^2 - r^2}{R_2^2 - R_1^2}$$

Taking various values of r, the corresponding values of f and p are plotted on Fig. 84. The working is given in Example 5 below. It will be seen that the circumferential stress f, which is the most important to be considered in this connection, varies from a maximum value on the inside surface to a minimum on the outside.

This means that the material on the inside is doing more than its share in withstanding tension; and, in order to equalise the tension stress under the maximum pressure in such cylinders as guns, it is usual to impose an initial compressive stress in the material. This stress is greatest on the inside surface, and

diminishes towards the outside, the result being that when the internal pressure is applied, the algebraic sum of the initial and

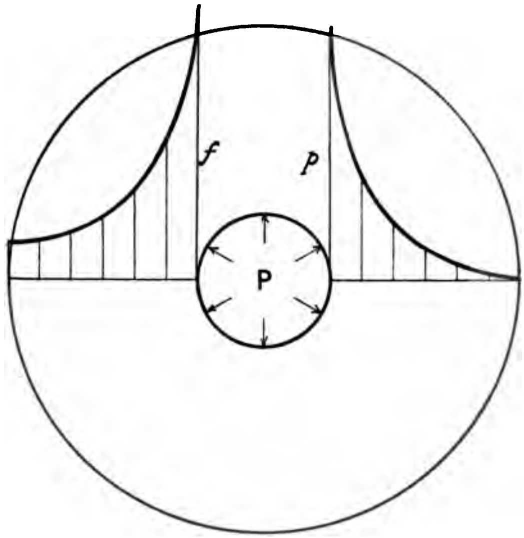

FIG. 84.

induced stresses is more nearly uniform than would otherwise be the case.

In guns this initial compressive stress is attained by winding successive layers of wire round the inner tube, the wire being kept under a uniform tension, or one which has a predetermined variation. This effect is shown in Fig. 85; here the initial compression stresses are plotted upwards and tensile stresses downwards. The stresses indicated by the curves are there due to the winding alone, before there is any internal pressure.

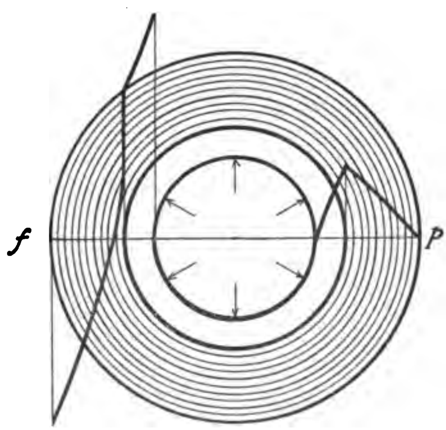

FIG. 85.

On Fig. 86 is shown how the initial stress due to cooling of the metal in cast cylinders varies from point to point. The metal on the inside and on the outside cools first, and as the intermediate part cools it shrinks, putting itself in circumferential tension, while the inner and outer parts are in compression.

It has been shown that

$$\frac{P}{F} = \frac{R_2^2 - R_1^2}{R_2^2 + R_1^2}$$

and from this it follows that if F be the ultimate tensile stress of a brittle material like cast iron, or the yield stress of a more ductile material, then when P is made = F, the R_2 must be infinitely great. In other words, it is impossible in a thick cylinder with no initial stress to use a pressure which is greater than F, however thick the walls may be.

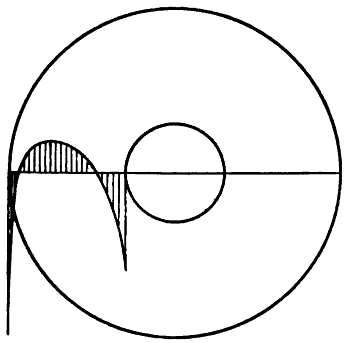

Fig. 86.

If this were attempted, failure would take place by cracks beginning on the inside and extending outwards.

Example 3.—A thick cylinder is built up so that the initial tensile stress of the outer and the compressive stress of the inner skin are both 3 tons per sq. in. Calculate the resultant stress of both skins when under internal fluid pressure of $4\frac{1}{2}$ tons per sq. in. Diameter, 20 ins. external and 10 ins. internal.

Considering cylinder without initial stresses,

$$f_1 = \text{stress on inner skin} = P\left(\frac{R_2^2 + R_1^2}{R_2^2 - R_1^2}\right)$$

$$= \frac{9}{2}\,\frac{5}{3} = \frac{15}{2} = 7\cdot5 \text{ tons per sq. in.}$$

$$f_0 = \text{stress on outer skin} = \frac{PR_1^2}{R_2^2}\left(\frac{R_2^2 + R_1^2}{R_2^2 - R_1^2}\right)$$

$$= 7\cdot5 \times \tfrac{1}{4} = 1\tfrac{7}{8} \text{ tons per sq. in.}$$

Hence resultant stress on outer skin

$$= 1\tfrac{7}{8} + 3 = 4\tfrac{7}{8} \text{ tons per sq. in.}$$

Resultant stress on inner skin

$$= 7\tfrac{1}{2} - 3 = 4\tfrac{1}{2} \text{ tons per sq. in.}$$

Example 4.—Find maximum and minimum stresses in the walls of a thick cylinder; internal diameter 8 ins. and external diameter 14 ins. Internal fluid pressure, 2000 lbs. per sq. in.

Maximum stress is at inner radius, and $= P\dfrac{R_2^2+R_1^2}{R_2^2-R_1^2}$

$$= 2000 \times \frac{49+16}{49-16} = 2000\ \frac{65}{33} = 3940 \text{ lbs. per sq. in.}$$

Minimum stress is at outer radius, and $= \dfrac{PR_1^2}{R_2^2}\left(\dfrac{R_2^2+R_1^2}{R_2^2-R_1^2}\right)$

$$= \frac{130{,}000}{33}\ \frac{16}{49} = 1286 \text{ lbs. per sq. in.}$$

Example 5.—In a cylinder where $R_1=1$, $R_2=4$, find the values of f at radii increasing by half-inches. $P=1000$ lbs., and plot the curve (Fig. 84).

At inside radius

$$f = \frac{P(R_2^2+R_1^2)}{R_2^2-R_1^2} = \frac{1000(16+1)}{16-1} = \frac{1000\times 17}{15} = 1133 \text{ lbs.}$$

At any other radius, r,

$$f = \frac{PR_1^2}{R_2^2-R_1^2}\left(1+\frac{R_2^2}{r^2}\right)$$

At radius 1·5 ins. $f = \dfrac{1000\times 1}{15} \times 1 + \dfrac{16}{2\cdot 25}$

			$= 66\cdot6\times 8\cdot1$	$= 539\cdot5$ lbs.	
„	2	„	$f = 66\cdot6\times 5$	$= 339$	„
„	2·5	„	$f = 66\cdot6\times 3\cdot56$	$= 237$	„
„	3	„	$f = 66\cdot6\times 2\cdot7$	$= 180$	„
„	3·5	„	$f = 66\cdot6\times 2\cdot3$	$= 153$	„
„	4	„	$f = 66\cdot6\times 2$	$= 133$	„

or at outside radius $f = \dfrac{2PR_1^2}{R_2^2-R_1^2}$

$$= \frac{2\times 1000\times 1}{15} = \frac{2000}{15} = 133 \text{ lbs.}$$

Also, using formula, $p = \frac{PR_1^2}{r^2} \frac{R_2^2 - r^2}{R_2^2 - R_1^2}$, the values of the radial stresses are found as follow:—

At radius	1 in.	p =	1000	lbs.	per sq. in.
"	1·5 ins.	p =	408	"	"
"	2 "	p =	200	"	"
"	2·5 "	p =	104	"	"
"	3 "	p =	52	"	"
"	3·5 "	p =	21	"	"
"	4 "	p =	0	"	"

Strength of Thin and Thick Spherical Shells.—Using the same notations as before, for thin shells

$$\frac{p}{f} = \frac{2t}{r} = \frac{4t}{d}$$

And for thick shells

$$\frac{P}{F} = \frac{2R_2^3 - 2R_1^3}{R_2^3 + 2R_1^3}$$

CHAPTER XII

RIVETED JOINTS

THE full discussion of the subject of riveted joints in all its detail really comes under the head of "Machine Design," but it may not be out of place here to notice some of the chief problems so far as they relate to the strength properties of the materials used.

Of the many forms of joint employed for connecting together the edges of iron and steel plates, those illustrated on the following sketches may be taken as among the more usual.

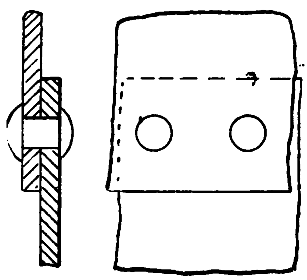

FIG. 87.

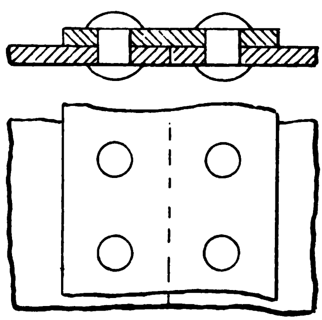

FIG. 88.

The single riveted lap-joint shown on Fig. 87 is the simplest. The edges of the two plates overlap, and there is one row of rivets which pass through the two plates.

Fig. 88 represents a single riveted butt-joint, where the main plates come edge to edge, and each is connected to an auxiliary plate, called a *cover plate*, by a single row of rivets.

The double riveted lap-joint on Fig. 89 has two rows of rivets arranged diagonally or zigzag.

On Fig. 90 is shown a similar joint to that on Fig. 88, but with double riveting and two cover plates.

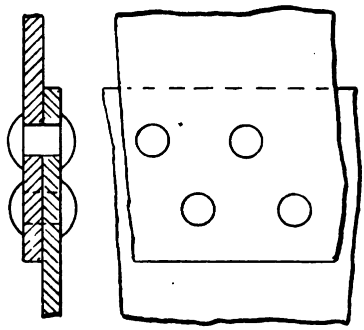

Fig. 89.

The usual theory upon which the proportions of riveted joints are made to depend is somewhat as follows:—

It is assumed that a joint may fail in one of five different

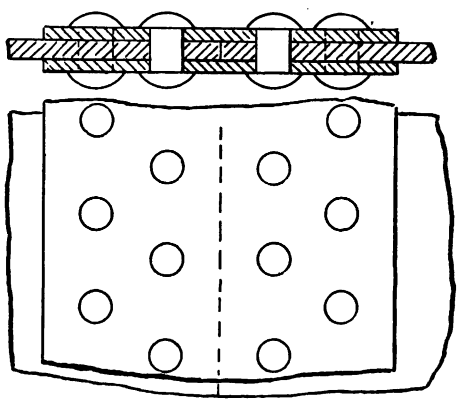

Fig. 90.

ways. Considering the simplest case of a strip of the plate joined by a single rivet, these are (Fig. 91):—

(*a*) By the shearing of the rivet.

(*b*) By the tearing of the plate through the rivet hole.

(*c*) By the rivet breaking through the plate in front of it.

(*d*) By the rivet crushing the material of the plate in front of it.

(*e*) By the rivet shearing out that part of the plate which is in front of it.

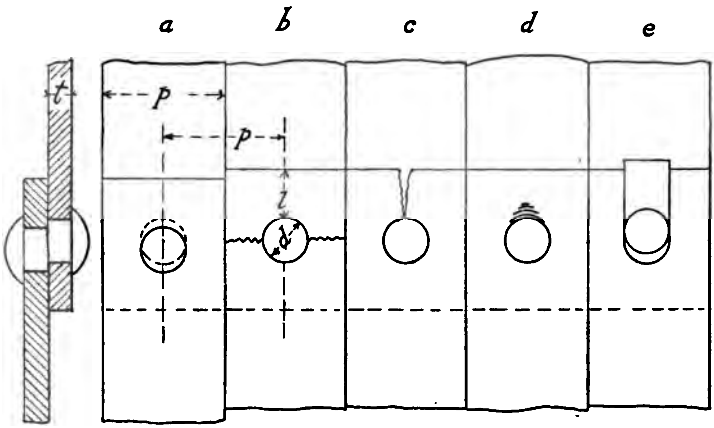

FIG. 91.

Now, calling

p = the pitch; that is, the distance from centre to centre measured along the line of the rivets; this will be the same as the width of the strip;
t = the thickness of the plate;
d = the diameter of the rivet;
l = the lap of the plate; that is, the distance from the edge of the rivet hole to the edge of the plate;
f_t = the tensile resistance of the plate;
f_s = the shearing resistance of the rivet;
f_c = the crushing strength of the plate;

then the resistance of the joint, in Fig. 91, to failure in each of the five ways enumerated above, will be:

(*a*) Resistance to shearing of rivet $= f_s \frac{\pi}{4} d^2$

(*b*) Resistance to tearing of plate $= f_t t(p-d)$

(*c*) Resistance to breaking through plate $= C\frac{tl^2}{d}$

This is a case of a short beam, fixed at the ends, and of span d, depth l, and breadth t, C being a constant depending on the material.

The value of C is given by some authorities as about 45 tons.

(*d*) Resistance of plate to crushing $= f_c td$

(*e*) Resistance of plate to shearing $= 2f_s t\left(\frac{d}{2}+l\right)$

An economically designed joint should be as ready to fail in one way as another; that is—

$$f_s\frac{\pi}{4}d^2 = f_t(p-d) = \mathrm{C}\frac{tl^2}{d} = f_c td = 2f_s t\left(\frac{d}{2}+l\right)$$

Generally speaking, (*d*) and (*e*) are not considered, as it is found that if the joint designed according to the usual rules is strong enough to resist (*a*), (*b*), and (*c*), it will be sufficiently strong for the other two.

Assuming that t is given, the diameter of the rivets must first be found. To do this, it is found that the following empirical rule gives the most satisfactory results for plates of the more usual sizes:—

$$d = 1{\cdot}25\sqrt{t} \text{ for boiler work}$$

$$\text{or} \quad d = 1{\cdot}1\sqrt{t} \text{ for bridge work}$$

Now, taking the former of these values, and equating (*b*) and (*a*),

$$f_t t(p-d) = f_s\frac{\pi}{4}d^2$$

$$\text{But} \quad d^2 = (1{\cdot}25)^2 t$$

$$\text{and} \quad f_s = 0{\cdot}8 f_t, \text{ very nearly}$$

$$p-d = 0{\cdot}8\times\frac{\pi}{4}\times(1{\cdot}25)^2$$

$$\text{or} \quad p = d+0{\cdot}98 \text{ in.}$$

This is from purely theoretical considerations, but it is found from experiment that the pitch must be somewhat greater, especially in the case of punched holes, where the metal of the plate is damaged by the punching for a small distance round the hole.

The following values, based on the experiments on riveted

joints carried out by the Institution of Mechanical Engineers, may be used:—

For iron plates and rivets	Punched . .	$p = d + 1{\cdot}50$ ins.
	Drilled . .	$p = d + 1{\cdot}40$ ins.
For steel plates and rivets	Punched . .	$p = d + 1{\cdot}13$ ins.
	Drilled . .	$p = d + 1$ in.

To find the overlap, l, equate (c) to (b).

$$C\frac{tl^2}{d} = f_s\frac{\pi}{4}d^2$$

$$l = \sqrt{\frac{\frac{f_s}{C}\frac{\pi}{4}d^3}{t}}$$

$$= 0{\cdot}8\sqrt{d}$$

Practical conditions, however, would seem to make it necessary to increase this to about $1{\cdot}1\sqrt{d}$. This means that, instead of (c) being equal in strength to (b), it is so far stronger that there is not likely to be any possibility of failure taking place in this direction.

It is important to remember that the pitch is partly dependent on the resistance of the rivet material to shear. In most cases the rivets are made from a material which is softer and more ductile than that used for the plates.

For a given ductile material the shearing strength may be taken as approximately equal to four-fifths of the tensile strength.

Efficiency of Riveted Joints.—The tearing resistance of the plate on a section taken through the centres of the holes is $f_t t(p-d)$, and the tensile resistance of the same length of the original plate is $f_t tp$. The former of these is obviously less than the latter, on account of the metal being cut away in making the rivet holes. For any given joint the ratio of one of these to the other is called the *efficiency of the joint*. In other words, the efficiency is the fraction of the strength of the plate which the joint provides. Theoretically, the efficiency is $\frac{f_t t(p-d)}{f_t tp}$ or $\frac{p-d}{p}$, but it is actually found that the efficiency is somewhat less for the reasons already mentioned as to the tensile strength per square inch of the drilled or punched plate being less than that of the uncut plate. If some

of these efficiencies are calculated it will be found that they are greater for the smaller sizes of plates, and that, for a given size of plate, the efficiency is greater where there are two rows of rivets than where there is only one, and still greater for treble riveted joints.

The following figures will serve to give an approximate idea of the values to be expected:—

EFFICIENCIES OF DIFFERENT RIVETED JOINTS.

	Punched Holes.			Drilled Holes.		
Size of plates	$\frac{1}{2}$ in.	$\frac{3}{4}$ in.	1 in.	$\frac{1}{2}$ in.	$\frac{3}{4}$ in.	1 in.
Single Riveting—						
Iron plates—Iron rivets .	56	51	48	59	54	50
Steel plates—Steel rivets .	51	47	44	54	49	46
Double Riveting—						
Iron plates—Iron rivets .	72	68	65	74	70	66
Steel plates—Steel rivets .	68	64	61	70	66	63
Treble Riveting—						
Steel plates—Steel rivets .	...	...	...	78	74	71

The above efficiencies are all given as percentages.

CHAPTER XIII

STRENGTH OF MATERIALS AS FOUND FROM THE RESULTS OF TESTS

In designing any kind of engineering structure the form and dimensions are made such that the loads which are likely to come upon it will not give rise to more than a certain predetermined stress in any part. In order to decide upon the amount of stress which may be allowed upon any given material when loaded in some particular manner, the strength of the material must be determined by experiment. An ideal kind of experiment which might be made upon a structure would be to test the structure as a whole with loads which are as nearly the actual loads as possible, and afterwards to carry them beyond their working values and see what happened. But it is not often that full-sized pieces can be tested, partly because the making of an actual structural piece which is to be immediately destroyed is costly, and, what is more important, there are few testing machines which will accommodate full-sized pieces, except very small ones. Testing of this kind is, in spite of the difficulties in the way, sometimes carried out, and always yields valuable results.

Testing.—By testing is generally meant the loading in a testing machine of samples of the material, either specially prepared or cut from the material which is being used. When in the testing machine observations are taken for the purpose of finding out what happens to the material when under the load.

These tests may be carried out, first, for the purpose of ascertaining new facts about a given material, as to its behaviour under certain given conditions, which is really a form of research; and, in the second place, with the idea of ascertaining whether material supplied by a maker possesses all the qualities which have been specified by the user. These last are commercial tests.

Specimens of the metals may be tested in tension, compression, shear, bending, and torsion; but, except in the case of cast iron, it is generally sufficient to make the first of these tests for the purpose of ascertaining the quality of the material. The reason is probably that the tensile test is the simplest and most easy to carry out, and because it leads to some five or six different quantities, each of which is capable of providing information regarding some different quality in the material.

In a tension test a sample of the material, generally in the form of a bar, is taken hold of at each end and pulled with an increasing amount of force. The function of the testing machine is to apply this pull at the will of the experimenter and at the same time to indicate its magnitude.

The observations usually made during the application of load to a test bar are, in addition to the magnitude of the load itself, the following:—

1. A series of measurements of the deformations accompanying the loads may be taken by using an extensometer, so as to make it possible for the modulus of elasticity to be calculated. This is rarely done in commercial work, though there may be occasions when an exact determination is required to satisfy a specification.
2. Such observations are taken as are necessary to fix the *yield point* load. Something has already been said about the elastic limit, proportional limit, and the yield point, and this will be again discussed more fully. For the present it is sufficient to say that the load on the specimen is increased until the yield point has been reached, and is then noted. The yield point may be found by using dividers, and seeing when a certain marked length of the bar begins to rapidly increase without any increase in load; or by noting the point when the weigh beam of the testing machine is seen to drop quickly on to its lower stop owing to the rapid stretch of the material.
3. After the yield point is passed the load is still further increased until fracture takes place, and the load which is required to bring about rupture is noted. This is called the *maximum load*, or, in some cases, the

ultimate load. The yield point load and the maximum load are divided by the original cross-sectional area of the bar to obtain the *yield point stress* and the *maximum stress.*

This last result is generally the one to which most importance is attached.

4. The *permanent elongation* of the material, as measured upon a length marked on the bar before the test, is also measured and noted. This may be upon a length of 10 ins., 8 ins., or any other length down to 2 ins., according to the size of the bar. The result is given as a percentage.
5. The amount the cross-section of the bar has been reduced is measured at the point of fracture, and is called the *reduction* or *contraction of area,* and is expressed as a percentage of the original area.

Nos. 4 and 5 yield information as to the ductility of the material.

The results usually required in specifications and by inspectors of material are :—

The yield point stress;
The maximum stress; and
The ductility, as indicated by the permanent elongation after fracture.

The two last are always insisted upon in tests of ductile material such as wrought iron, mild steel, and bronze. The yield point stress and the reduction in area are less frequently required.

The principal forms of test bars used for ductile materials are indicated by the sketches on Fig. 92.

Here, A is a round bar whose middle portion has been turned to a smaller diameter.
B is a plain round or square bar.
C is a short turned specimen with screwed ends, such as might be cut out of a large forging.
D is a plate or flat bar specimen with the middle part milled out to a smaller width.
E is a similar bar untouched.

F is the kind of specimen used for cast iron, the centre part being turned and the ends formed to fit spherical dies.

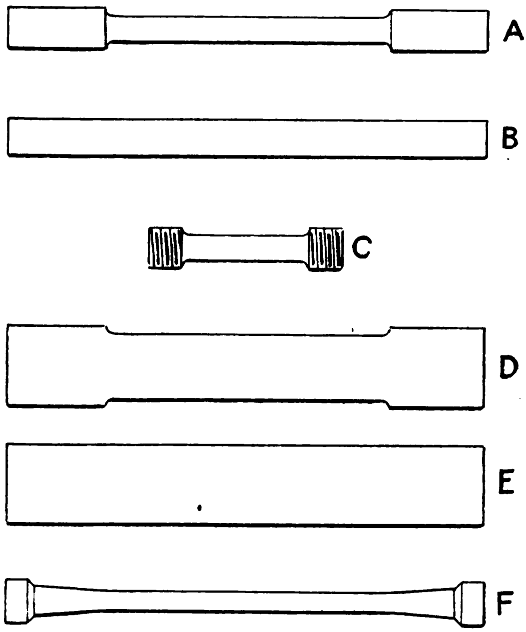

Fig. 92.

Testing Machines.—The essential points of any testing machine for carrying out the tests which have been and which will be mentioned, are :—

(*a*) Some kind of device for holding the ends of the specimen in such a manner that they cannot be pulled away by the load which is applied. This may be in the form of steel wedges with serrated surfaces, or the ends of the bar may be screwed so as to fit into corresponding sockets in the jaws of the machine.

(*b*) There must be some way of applying any desired load to the bar.

(*c*) The last, and what is the most important part of the machine, is the arrangement for measuring the magnitude of the applied load.

(*d*) It is also necessary that the grips at one end of the specimen should be capable of being moved at the will of the experimenter, so as to enable him to take up any stretch which may occur in the bar during the test.

A skeleton drawing of a simple form of testing machine is given on Fig. 93. This represents the vertical type made by Messrs Buckton of Leeds.

Here the specimen is at A and is held in a vertical position, its lower end being attached to a screw by means of which it is pulled downwards as desired and a tension put upon the bar. The upper end of the specimen is connected to the short arm ot a lever or

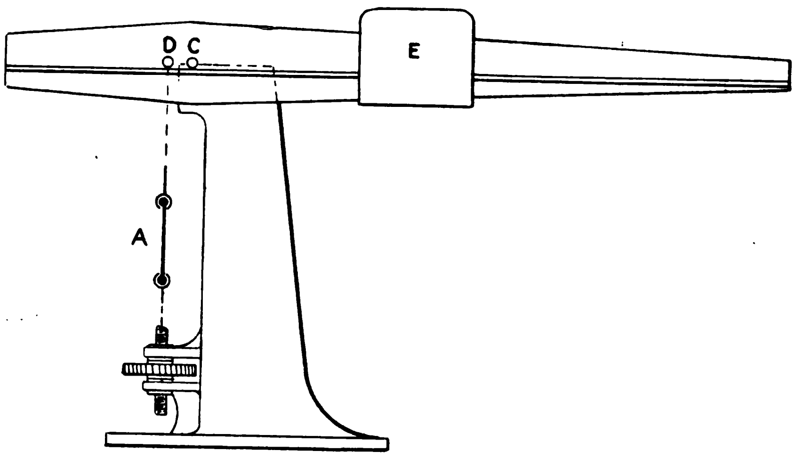

Fig. 93.

steelyard at the point D; this lever turns about a fulcrum C, its other end being pulled downwards by a moving weight E. The amount of the tension on the specimen depends upon the magnitude of the weight E, and, at the same time, upon the ratio of EC to DC. Most machines in use in this country are made to depend upon this plan of having a screw or hydraulic device by which the load is applied and a weighing lever for the purpose of measuring this load. Where it is desired to have the specimen in a horizontal instead of a vertical position, a knee-lever is interposed between the weighing lever and the specimen.

For compression the links suspended from D are fixed to a

plate, and the links from the screw are attached to another plate, this being pulled downwards by the screw towards the first plate, which is placed below it.

In another type of testing machine which is coming into use in this country, the load is applied by the pressure of some fluid such as oil acting on a plunger or ram, and the magnitude of the load is measured by the pressure of the fluid. So long as there is no appreciable friction in the ram, this fluid pressure will be proportional to a measure of the load. For a more detailed description of the several forms of testing machines the reader is referred to one of the books which deal exclusively with the subject.*

Appliances for Measuring Elastic Deformations.—Where measurements for the purpose of calculating Young's Modulus have to be made, some form of extensometer must be applied to the specimen under observation. The mirror instrument devised by Professor Martens of Charlottenberg has already been described in Chap. VII.

Besides the sketch on Fig. 61, which explains the principle of action of the Martens mirror apparatus, views are given on Figs. 94 and 95.† These are taken from photographs of the apparatus, when actually applied to a vertical bar. The bar shown here is one used by the writer for the purpose of explaining the action of the apparatus to students, and for enabling them to obtain some little practice with it before using it on a specimen under stress. It is made telescopic, so that its total length can be varied at will by simply turning the milled head of a differential screw. The appliance can be put upon any table or convenient stand, and, besides being applicable for demonstration purposes, it is found extremely useful when an extensometer is to be calibrated or tested, or when two extensometers are to be compared. The mirrors are shown more in detail on Fig. 96.

There are two measuring bars, one on the surface of the bar next the telescope, and the other on the surface farther from it. By adopting this arrangement the observer can see how the stress varies at the different parts of the section. This is not possible

* See the Author's book on *Testing of Materials*, published by the Scientific Publishing Co., Manchester.

† The photographs on Figs. 94 and 95 were kindly taken in the photographic department of the Manchester School of Technology.

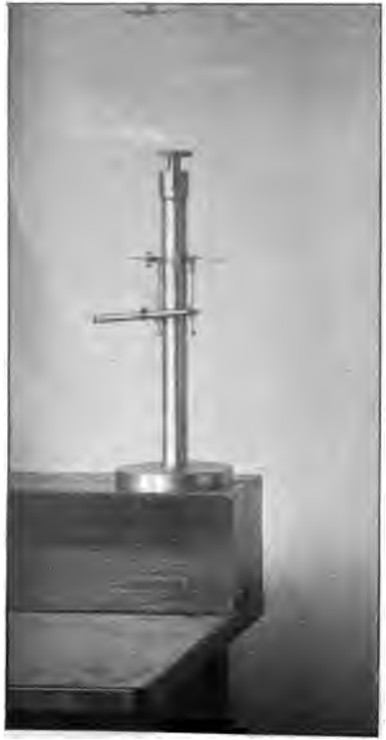
FIG. 94.


FIG. 95.

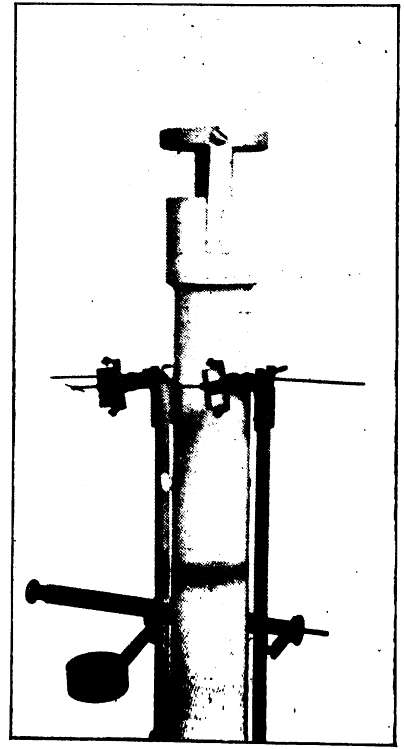
FIG. 96.

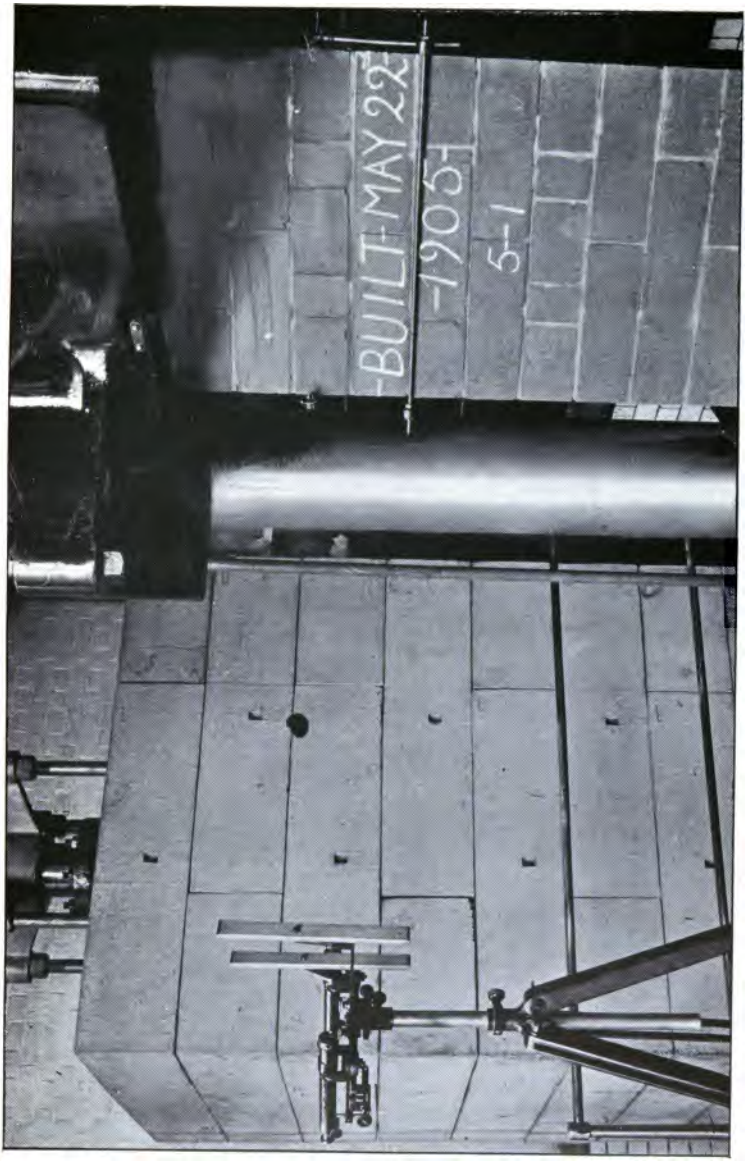

FIG. 97.

[*Between pp.* 136 *and* 137.

with most extensometers, where only the average strain is measured and the observer cannot tell how nearly his load is being applied along the axis of the specimen. In the Martens mirror apparatus a number of measuring bars are usually supplied of different lengths, some for flat and some for rounded specimens.

This instrument is very useful in that it can be just as readily applied to actual parts of machines or structures under stress as to test specimens. On Fig. 97 it is shown when being used for measuring the compressions of a brickwork pier.

Its fixing and adjustment is sometimes a little tedious at first, but when once set it is found to give extremely good results.

Among the other extensometers in use in this country is that of Ewing, in which a micrometer-microscope is combined with a lever; and the simple lever instruments of the Kennedy type, where the change in length of the specimen moves the short arm of a lever with a high velocity ratio, the outer end of the long arm carrying a pointer which moves over a graduated scale.

Where material is tested under a compressive stress, it is generally necessary to bed the specimen in some substance such as Portland cement, plaster of Paris, lead, or millboard, in order to distribute the stress uniformly over the surface. This is found to be necessary in the case of such substances as stone, brick, cement, and concrete; where the specimen is of metal, it is usually possible to machine the surface so as to get a flat surface to rest against the compression plates of the machine.

Besides tensile and compressive tests, bending or cross-breaking tests are frequently resorted to. These are used in the case of timber beams, iron and steel girders, rails, and so forth.

The torsional strength of small shafts can be found experimentally by applying a torsional moment in a special machine; and direct shearing tests are also carried out for some purposes.

There is little difficulty in determining and interpreting the maximum stress of materials tested to destruction and the elongation and reduction after fracture, but it is found that, as regards the limits of elasticity, there is sometimes considerable ambiguity; this point will be discussed in the following chapter.

CHAPTER XIV

THE LIMITS OF ELASTICITY

It has already been pointed out that when a bar is loaded it is altered in length, and that if when the load is taken off the bar returns to its original dimensions, it is said to be perfectly elastic. It has further been shown that as the loading is increased, there comes a point beyond which the change in length is partly temporary and partly permanent, and that the stress at which this begins to take place is the elastic limit of the material. This is the simplest, and in many ways the most scientific description of the elastic limit; but, as will be seen, there are great difficulties in the way of determining this point, and its very existence depends upon the precision of the instruments used in its determination.

Limit of Proportionality.—It is found from experiment that the proportionality between load and deformation or stress and strain exists until a stress is reached beyond which the strains begin to increase for equal increments of stress. Take the case of a bar under tension in a testing machine: the loads are applied so much at a time, the increase being the same at every step. Means are taken for measuring the small extensions accompanying the tension stresses, and it is found that for every equal increase of load a correspondingly equal increase of length takes place. This goes on until a certain load is reached, beyond which the increments of stretch are no longer equal, but continually increase. This point has been given the name of *limit of proportionality*, or *P-limit*. This term clearly defines its own meaning, and could not be improved upon, but, possibly with some reason, it has often been called the elastic limit.

So far, it is seen that there are two ways in which the elastic limit stress may be defined—viz., as the point beyond which permanent

set is given to the bar, and otherwise as the stress beyond which the strain is no longer proportional to the stress. If these two points were found to coincide, there would be no difficulty about the elastic limit; but, unfortunately, it is usually found that the first permanent set takes place at a stress below that at which proportionality appears to cease. It is therefore necessary to clearly distinguish between these two; they are so often confused, and hastily assumed to be one and the same point. If the original term, *elastic limit,* is retained for the first—and that would seem to be the most reasonable course to adopt—then the second can still be called *P-limit.*

It has been said that, using the most precise and accurate measuring instruments at present constructed, the elastic limit is found to lie below the P-limit; but the opinion is strongly held by some authorities—notably M. Fremont—that the elastic limit and the P-limit are coincident, and that it is only want of accuracy and precision in our measuring instruments which makes it appear that they occupy distinctly different positions.

In his investigations into the position of the elastic limit of metals, M. Fremont has adopted the method of using a microscope to examine the polished surface of the metal under stress and has taken as the elastic limit a point at which a slight dulling of the surface is shown by the microscope; and he maintains that this is the only true elastic limit. In the course of his examination he found that this limit only appeared at first in one or more isolated spots, until the change gradually covered the whole of the surface; and in order to overcome the difficulty of noting the first appearance he used bars of gradually varying section, in which there was one small part where the stress was always greatest, and at which the appearance would change as soon as the stress at the smallest area reached the elastic limit stress.

As a result of these investigations of M. Fremont, a third and new definition of the elastic limit is created, which only tends to add to the already existing confusion as to what really fixes this point.

Besides these, there have at different times been other attempts to solve the difficulty. Of these, Styffe wanted to fix the limit as a point which was made to depend on the increase of permanent strain as depending on the rate of increase of stress. By this method a point is obtained for the limit which lies above

the P-limit. In a similar manner Wertheim attempted to fix the limit as a stress where the permanent strain was a definite fraction of the original length of the specimen, namely $\frac{1}{20000}$. There are many objections to this plan, chiefly on account of the delicacy of the operation and the great care required in carrying it out, and also because only part of this apparently permanent strain is really permanent, the remaining portion disappearing after the bar has rested for a time.

Yield Point.—The five points which have been mentioned as elastic limits are all more or less entitled to some claim on the term, but this does not apply to the point so often mentioned in commercial tests as the elastic limit, but which is really the yield point. This is the stress, readily found, where the strain rapidly increases in amount as the load is slowly increased or is stationary, and is clearly discernible by means of compass measurements and by the appearance of the surface of the bar. The various ways of fixing this point include the dropping of the testing-machine beam on to its bottom stop, advocated by Kennedy; the change in the appearance of the surface of the bar recommended by Styffe; the perceptible thickening of a line inscribed on the bar by a pair of compasses, which method is used by many experimenters in this country; and the employment of an autographic diagram, also much used and recommended by Unwin. A way of finding the yield point which the author has often employed is to use an extensometer and note the load at which the pointer or cross-wire begins to creep quickly along the scale without any increase in load. This, though probably the most reliable method, is somewhat elaborate, and requires more time than can be spared in ordinary commercial testing operations.

Illustrations of the Three Limits.—It will be well here to inspect one or two diagrams which serve to illustrate what has been said regarding the several points which have been mentioned as elastic limits.

The curves on Fig. 98 have been plotted from the results of a tension test on a turned bar of wrought iron, the measurements of extension being taken with a Martens mirror extensometer capable of reading directly to $\frac{1}{10000}$th of an inch, and by estimation to $\frac{1}{50000}$th of an inch. The extensions were measured on a 10 cm. length of the bar.

The wrought iron bar in question was put into the testing machine, and loads applied by increments of 2 tons. Before each new load was applied the load was removed, and the readings on the extensometer scales, if any, taken. In this way it was possible to determine the point at which the reading line did not return to zero after loading—the reading at zero load

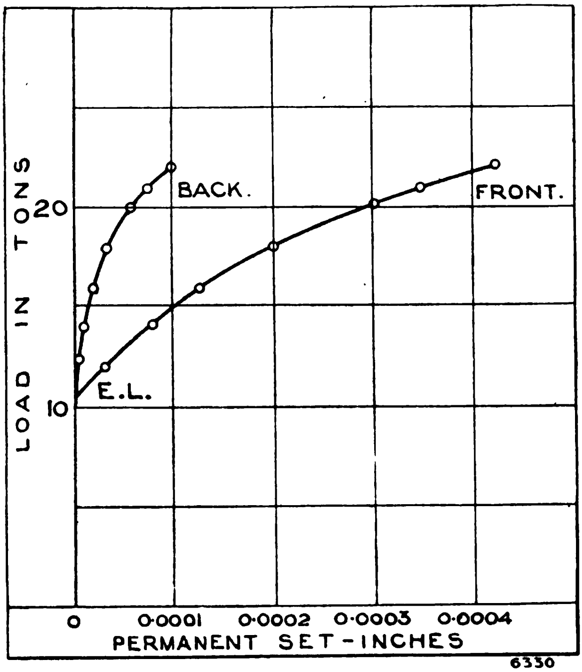

FIG. 1.—BAR OF WROUGHT IRON IN TENSION.
DIAM., 1·62 IN. MEASURED LENGTH, 4 INS.

Fig. 98.

giving the permanent set produced by the last loading. By doing this, it was possible to determine, with a fair degree of accuracy, at what load the bar first began to take permanent set.

The curves on Fig. 98 represent the permanent set for each increment of 2 tons up to 22 tons load. Here loads are plotted vertically and extensions horizontally. It will be seen that until about 12 tons have been reached there is no permanent set, the reading line of the extensometer coming back to zero after each

loading. At this load a permanent set is first noticed, and increases for each successive load. The two curves represent the readings on the two sides of the bar, one being marked on the diagram as "back," the other as "front." It will be seen that the permanent set, besides commencing at a definite point, increases by regular amounts, and also that this increase is more rapid on the front of the bar than on the back. From this it is gathered that the bar is not held evenly in the grips, so that the line of pull lies between the axis of the bar and its front side, causing the stress on this side to be greater than at the back.

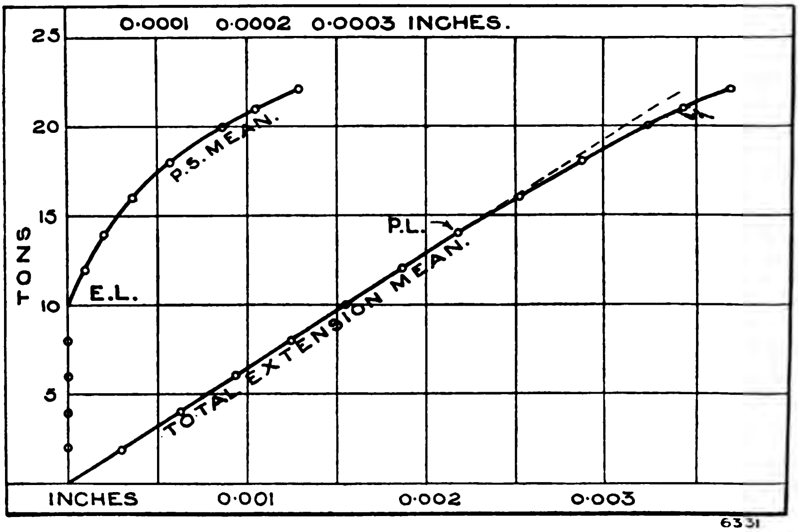

Fig. 99.

The point where the lines first leave the vertical zero line is marked on the diagram E.L., being the load at which the elastic limit, as above defined, has been reached.

Now look at Fig. 99. This refers to the same test as the last. Two curves have been plotted, the upper marked "P.S. mean," and is an average of the two plotted on Fig. 98, again showing the elastic limit. On the other curve have been plotted the total average extensions. It will be seen that up to a certain load the points of the curve fall on a perfectly straight line, thus indicating true proportionality between load and extension, or

between stress and strain. This point where the curve leaves the straight and begins to turn slightly towards the right, is marked P.L., indicating the P-limit, or limit of proportionality. This point occurs at a load higher than the elastic limit just found.

This illustrates what has previously been said about the relative positions of the elastic limit and the P-limit, namely, that the latter is found to be higher than the former; but it is quite conceivable that, as M. Fremont considers likely, these two points really coincide, and would be found to coincide if the measurements of strains could be made with greater precision than is now the case. The way in which the proportionality limit is usually found, and this is probably the most reliable way, is to measure the strains corresponding to known stresses, and plot one against the other to a fairly large scale on true squared paper. A straight-edge is then laid along the series of points so obtained. It will be found that the straight-edge can be made to coincide with the earlier part of the series of points, and it is clearly seen where the line begins to leave the straight. It would seem that this point of departure ought to occur precisely at the same load as that at which it was first found that there was any permanent set, and that the more nearly they would so coincide, the more precise were the strain measurements and the larger and more accurate the plotting. As a matter of fact, it is found to be difficult, with specimens of the ordinary sizes, even when estimating to fifty-thousandths of an inch, to make the increments of strain quite uniform. It is also difficult to obtain perfect uniformity in the increments of the load, and this leads to small unevennesses in the strains, apart from the intrinsic want of precision in their own readings. Taking all these considerations into account, the writer believes that the elastic limit ought to coincide with the P-limit. Whether greater precision in measuring would fix the elastic limit more accurately is open to doubt when the form of the set curve is examined: the curve appears to run into the line of zero strains at a definite angle, and not to approach it as to a tangent.

Next, referring to Fig. 100, it will be found that a stress strain diagram has been plotted which shows the P-limit and the yield point for the same bar. In this case the extensions were taken with a Ewing extensometer on $1\frac{1}{4}$ ins. of a tension specimen of

high-carbon steel $\frac{5}{8}$ of an inch in diameter. The diagram forms a straight line up to the proportionality limit, which is shown at 4 tons, corresponding to 13·06 tons per sq. in. Beyond this point the line diverges from the straight, curves more and more rapidly towards the right, and eventually becomes horizontal, in this way showing a rapid increase of length without any increase in load. This is the yield point, and is marked at 6 tons, or 19·74 tons per sq. in. on the diagram. It may be added that the maximum strength of this specimen was found to be about 47 tons per sq. in.

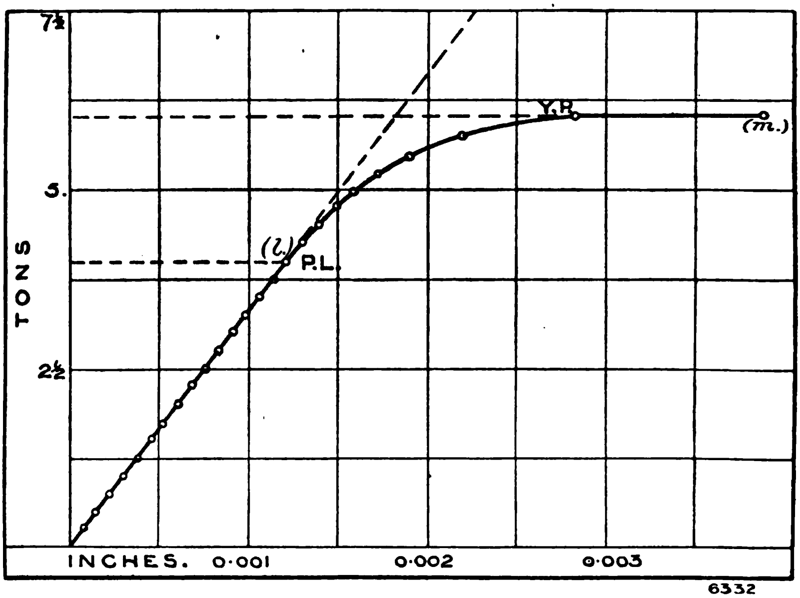

FIG. 100.

To anyone who is observing the extensometer readings, this yield point is determined with great certainty by noting when the reading line begins to creep rapidly along the scale without any increase in the load. It is not necessary to take a series of observations, but simply to watch the scale as the load is being gradually increased, and it is at once seen when the creeping begins. As already pointed out, for commercial purposes the use of an extensometer is generally found to occupy too much time, and the yield point is more usually found by using a pair of dividers or watching the drop of the beam, though the latter is

apt to be misleading in some cases. Another plan which has been mentioned is to let the bar tell its own tale by drawing an autographic diagram. Three such diagrams are shown on Fig. 101, the yield point being indicated in each case. In the curve taken from a mild steel bar (*a*), the position of the point is unmistakable, and is shown by the distinct horizontal jump in the curve where yielding takes place. It is to be noticed that the early part of this step is similar to the curved part of the diagram on Fig. 100, but drawn to a much smaller horizontal scale; the portion referred to is lettered (*l*) (*m*) in both diagrams.

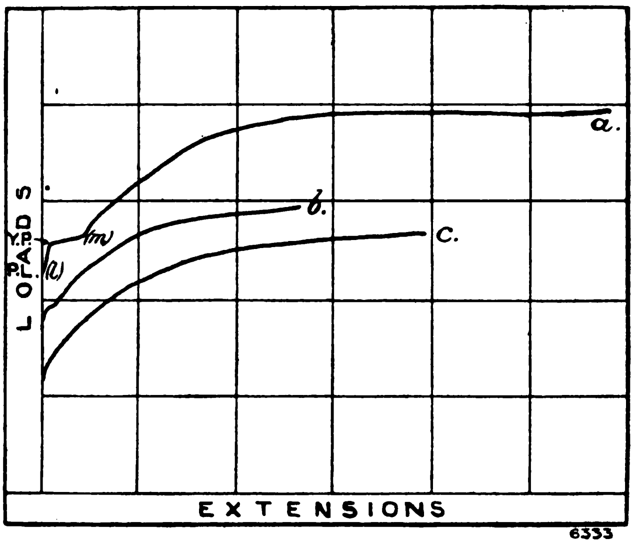

FIG. 101.

The second of these curves, (*b*), has a much less distinctly marked yield point, and the third one, (*c*), for a bar of wrought iron, is even worse. In cases like the last, the autographic diagram is apt to fail as a means of locating the yield point, unless great care is used in manipulating the apparatus and it is in the hands of a very experienced observer. In any case, the actual quantitative value of the yield point load should not be scaled from the diagram, but read off on the scale of the machine as soon as it is seen that the pencil is leaving the elastic line.

The writer has tried having three observers watching for the

yield point on the same test. One used an extensometer, the second tried the measured length of the bar with a pair of dividers, while the third watched for the drop of the beam. It was found that the extensometer man noted the point first, the dropping of the beam came second, and the dividers were last.

Unsymmetrical Loading.—It has already been pointed out that in very many cases the bar under test is loaded unsymmetrically, with the result that there is a greater stress on one side than on the other. The curves of extension for the two sides of

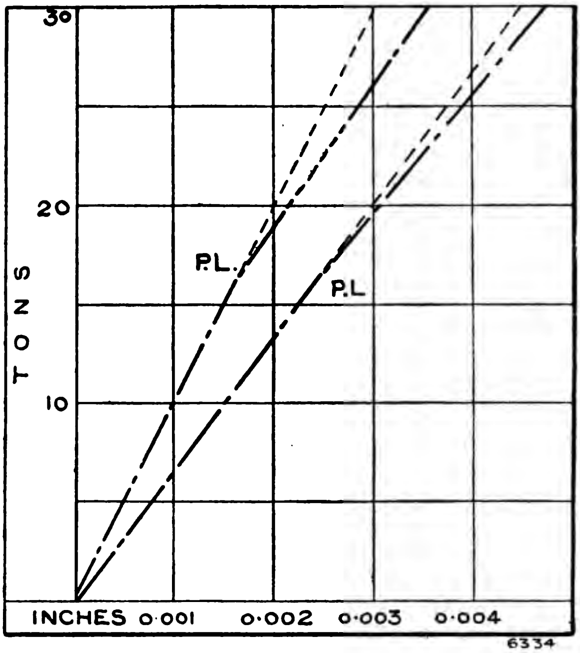

Fig. 102.

such an unequally loaded bar are shown in Fig. 102; this refers to a cast steel round bar, rather more than 2 sq. ins. in area. It will be seen that the two limits are not very far apart, that on the right-hand diagram occurring slightly earlier than in the case of the other. As the extensions, and consequently the stresses, of the right-hand side are greater than those for the left, as shown by the greater slope of the curve, the proportionality limit will occur first here. The effect of this slight yielding appears to throw the

load nearer the other side and bring the stress to a more nearly uniform condition, so that the proportionality limit is reached on the left side almost as soon as on the right. This approach to uniformity is shown on the diagram by the fact that the two dotted lines are nearly parallel. This would appear to show that where the loading is slightly eccentric, the general P-limit will only be very little in excess of the same point for the more greatly stressed portion of the bar.

The above are only isolated examples taken from a large number of similar experiments.

Changes of Limit by Previous Loading.—In what has been said so far, it has been assumed that the limit, whether elastic limit, P-limit, or yield point, is a fixed point for a given material; but this only so long as the material is always in precisely the same condition.

It is well known that if the P-limit has been found for a bar under a tension load, and the load is raised to a point somewhat in excess of the limit just determined, then on reloading the bar with the intention of finding the limit a second time, this will now be found to be at a higher stress, not far from the maximum stress applied in the first loading. In other words, the limit will have been raised by this previous loading, and not only so, but it may be raised time after time with successive loadings. Moreover, these artificially raised limits can be made to again fall to somewhere near the original point in several ways, such as resting for a time, by the application of heat, or by hammering. If, however, the bar has been previously loaded beyond the limit in tension, then the loading is reversed and the stress is carried beyond the limit in compression; it will be found that on now reloading in tension, the limit, instead of having been raised by the previous loading, will have been depressed. This appears to lead to the conclusion that the effect of previous loading beyond the limit is to raise the limit for loading in the same direction, and to depress it by loading with the same stress in the opposite direction, all these being artificially produced limits. Further, it is found that while an artificially raised limit is lowered by a reversed loading, an artificially depressed limit is raised by a reverse loading. The result is that after a series of reversed loadings, carried in each case just beyond the limit, the limit settles down to a fixed value which is the same for compression as for tension; this point has

been styled the *natural limit* for the material. The limit found on the first loading of the bar is generally higher than the natural limit, and has probably been artificially formed in the process of manufacture or by after-treatment. It has been found that bars return to their primitive limit after resting for a few weeks or months, or more quickly by the application of a temperature such as that of boiling water, or by hammering.

Looking back at what has just been said, it will be seen that there are really two well-defined limits found during the loading of a bar of iron or steel in tension. These may be called the limit of proportionality (or P-limit) and the yield point. The former of these is less easy to determine than the second, and its exact location partly depends on the precision of the measuring instruments used in its determination, and in all probability it coincides with the true elastic limit. It may safely be said that whenever the elastic limit is spoken of in connection with a commercial test, what is really meant is the yield point. Of course, if it is generally agreed to call this point the elastic limit, well and good, but this should be clearly stated, and the existing ambiguity removed once for all. The objections to this use of the term will be gathered from what has been said above; and, in addition, it is well known that many of the high-carbon steels show no yield point, and a limit can only be determined by using instruments of precision.

Besides defining what is meant by the limit given in commercial tests, it would be well if the manner of finding it could be clearly defined, and one standard method adopted. This would lead to uniformity, and enable useful comparisons to be made.

And lastly, where the limit given is to be in any sense used as a criterion of the quality of the metal, not only should the nature of the limit and its manner of determination be clearly understood, but the history of the test specimen should be known as regards its treatment between leaving the rolls and being tested.

The greater part of the matter in the above chapter is taken by permission from the author's article in the *Engineering Review*, April, 1904.

CHAPTER XV

THE MATERIALS USED IN CONSTRUCTION

THE materials of engineering naturally divide themselves into three chief classes, namely:

1. *Metals*, of which the most important are the several alloys of iron, copper, and steel.
2. *Vitreous materials*, including stones of various kinds, brick and terra-cotta, lime, cement, mortar, and concrete.
3. *Materials which do not come under the above heads*, such as timber, ropes, belting, and others of minor importance.

The above will be taken in the order given, and their chief strength properties, so far as they affect the engineer, will be briefly discussed.

Iron and Steel.

The metal **Iron**, alloyed with greater or lesser quantities of other elementary substances and containing various impurities, goes to form what are now among the most useful and important materials employed by the engineer.

There are a great number of these alloys of iron, ranging from the softest kind of wrought iron to the hardest steel, but they may be roughly classed under the following heads:—

Wrought Iron,
Mild or Low-carbon Steels,
Medium Steels,
High-carbon Steels,
Cast Iron.

The above are placed in the order of their percentage of carbon, wrought iron containing the lowest and cast iron the

highest. In preparing any of these varieties of iron, the material operated upon is pig-iron, the relatively crude form of iron which is obtained from the smelting of the ore. As the strength properties of these materials depend to a very large extent on the processes of manufacture, as well as upon the ingredients which are alloyed or mixed with the iron, the methods used in the principal operations will be very briefly indicated.

In addition to other impurities, the pig-iron of commerce contains carbon ($2\frac{1}{2}$ to 5 per cent.), silicon, and manganese. Wrought iron, on the other hand, contains little more than traces of these, so that the process in which wrought iron is prepared from pig consists essentially in burning the alloyed metals out of the iron, and leaving what is approximately pure iron. In the puddling process, by which much of the wrought iron is made, the pig-iron is brought to a molten condition and exposed to the action of a decarburising or oxidising flame in a reverberatory furnace. After a time the carbon, silicon, and manganese are burnt out of the mass, which has now become thick and sticky. This is worked up into a ball and placed between squeezers to get rid of the iron oxide or scale, and the billet so obtained is afterwards rolled into plates and bars. The rolling does not take place in one process, but is repeated in a series of operations. That is to say, in rolling bar iron, a number of the bars rolled from the original billet are bundled together, heated, and again rolled out into a single bar. This treatment results in the fibrous and laminated appearance seen in the fracture of a test specimen of wrought iron.

The result of the process of manufacture of wrought iron from pig is to get rid of almost all the carbon, as well as the greater part of all the other ingredients, leaving little more than traces of carbon, silicon, manganese, sulphur, phosphorus, and copper. The three last exist only as undesirable impurities.

The material having this composition, as produced by the puddling or the crucible process, and afterwards treated by hammering and rolling, is known as wrought iron, and has the following distinguishing qualities:—It is soft, tough, ductile, and fibrous in structure; it is equally strong in tension and in compression; it is capable of being easily pressed and hammered into various shapes when it has been softened by being brought to a red heat, and less readily when cold; two or more pieces can

be welded into one by hammering at a white heat; it is only at very high temperatures that wrought iron is reduced to a fluid state, thus rendering it unsuitable for castings; and, lastly, unlike the hard steels, it cannot be hardened by heating and quenching.

Of the ingredients included in wrought iron, carbon is generally less than 0·10 per cent., often as low as 0·03 per cent., and occasionally as high as 0·19 per cent.; of manganese there is little more than a trace; the silicon varies from a trace to 0·13 per cent.; the sulphur varies from nothing to 0·025 per cent.; and there is of phosphorus 0·02 to 0·20 per cent.

The effect of a deleterious quantity of phosphorus is to cause metal to be *cold short*, or to show a want of ductility when worked cold, and in a like manner sulphur and copper produce the same effect when the iron is being worked at a red heat, or makes it *red short.*

When tested under a *tensile* load it is found that the yield point of wrought iron is from 12 to 17 tons per sq. in., and the maximum stress is from 19 to 25 tons per sq. in. A good average value for the maximum stress is 23 tons per sq. in. The elongation after fracture varies from 10 to 30 per cent. and the reduction in area from 10 to 50 per cent. The appearance of the fracture should be fibrous, without showing too much lamination. A test bar broken under a steady load should not show a crystalline fracture.

In *compression* there is no maximum, as squeezing of the material can go on indefinitely. The definite point to be noted in the loading is the yield point, and this occurs at about the same stress as for the same material in tension.

The shearing strength is found to be from 75 to 85 per cent. of the tensile. A common value for this ratio used in design is 80 per cent., or four-fifths.

The elastic modulus for wrought iron is in the neighbourhood of 27,000,000 to 29,000,000 lbs. per sq. in.

It is found that wrought iron attains its maximum strength at a temperature of from 500° to 600° F.; after this temperature has been reached the strength falls off rapidly.

A typical stress strain diagram for wrought iron is shown on Fig. 103.

Steel.—Steel is iron containing from 0·05 to 1·50 per cent. of alloyed carbon, with the addition of small quantities of manganese,

silicon, sulphur, and phosphorus. The composition of low-carbon steel, used for such purposes as structural work and boilers, is somewhat as follows:—

Carbon	.	0·170 to 0·275 per cent.
Silicon	.	0·020 „ 0·090 „
Manganese	.	0·050 „ 0·700 „
Sulphur	.	0·010 „ 0·040 „
Phosphorus	.	0·010 „ 0·050 „

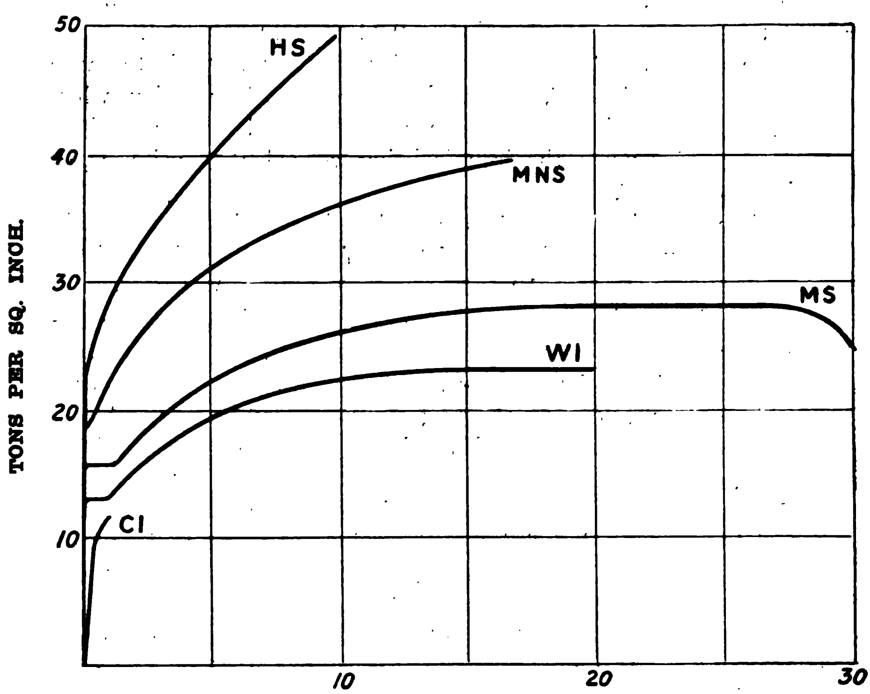

FIG. 103.—Here CI refers to cast iron; WI, wrought iron; MS, mild steel; MMS, medium steel; HS, high-carbon steel.

The Low-Carbon Steels are prepared from pig-iron, either by burning out the carbon and silicon and putting back sufficient carbon to give it the requisite qualities, as in the Bessemer process, or by melting pig-iron containing too large a proportion of carbon with a second ingredient which contains less carbon, such as scrap wrought iron or hæmatite iron ore, in such propor-

tions that the resulting material contains just the right percentage of carbon: this is what is done in the Siemens and Siemens-Martin open hearth processes. In all these methods the metal so produced is run into moulds so as to obtain ingots or slabs of the steel, and these are afterwards hammered and rolled into the desired bars or plates or section bars.

The general properties of the low-carbon or mild steels so produced are similar to those of wrought iron, with one or two modifications. The strength properties are higher, both as regards the yield point and maximum strength, and also the ductility as given by the elongation and reduction; the appearance of the fracture, instead of being fibrous, should have a fine silky or velvety appearance.

The yield point when under tensile stress varies from 13 to 22 tons per sq. in., and the maximum stress from 25 to 32 tons per sq. in., while the elongation is from 20 to 30 per cent., and the reduction in area from 40 to 60 per cent.

Like wrought iron, the compressive yield point stress of mild steel is the same as that in tension; and the shear strength is about four-fifths of the tensile.

Young's modulus generally falls between 28,000,000 and 30,000,000 lbs. per sq. in.

The maximum tensile strength occurs at 500° or 600° F.

A typical stress diagram is given on Fig. 103.

Medium Steels.—In what are known as the medium steels the percentage of carbon is higher, the strengths are higher, and the ductility somewhat less. These medium steels cover a considerable range as regards their strength properties. They are made by both the open hearth and Bessemer processes, and have the following compositions:—

Composition of Medium Steels.

Carbon . .	0·300 to 0·450	per cent.
Silicon . .	0·050 „ 0·060	„
Manganese .	0·450 „ 0·600	„
Sulphur . .	0·020 „ 0·040	„
Phosphorus .	0·030 „ 0·070	„

These medium steels are used for such purposes as engine forgings, piston rods, wheel tyres, rails, and any similar purposes

where a steel is required which is not very hard, but which is harder and stronger than ordinary low-carbon structural steel.

The yield point stress in tension of the medium steels varies from 14 to 22 tons per sq. in., and the maximum stress from 32 to 44 tons per sq. in.; while the elongation is from 12 to 20 per cent. The compressive and shearing strengths follow the same law as in mild steel. While the elastic modulus is very little higher, rarely being very far from 30,000,000 lbs. per sq. in.

It will be seen that the general properties of the medium steels are similar to those of the low-carbon steels, but in advance of them in the same way as mild steel is in advance of wrought iron.

High-Carbon or Hard Steels.—The percentage of carbon present in these steels varies from 0·45 per cent. to 1·50 per cent., and for some special purposes is still higher. Under this heading are included many varieties of steel produced in a number of ways and used for purposes where either great strength or great hardness is required. They include hard steel for forgings, steel for guns, springs, and every kind of cutting tool. In many of the purposes for which hard steel is used, it is necessary not only to have strength and hardness, but to be able to control the degree of this hardness. This is the one characteristic property possessed by the high-carbon steels which does not belong to wrought iron and the lower carbon steels. It is well known that this hardening can be brought about by heating the steel to redness, and cooling it quickly by plunging in water or oil. The degree of hardness depends upon the temperature of the steel when cooling takes place. It is usual with engineering tools to first make the steel very hard by quenching in cold water, and afterwards to let it down to the required "temper" by gradually raising the temperature until the required point is reached, and then again quenching in water. In the process of this reheating, the brightened surface of the metal changes colour, each tint corresponding to a certain degree of hardness.

The tensile strength of the hard steels may vary from 34 tons per sq. in. up to considerably over 100 tons per sq. in. Their ductility is not great in those steels with very high tensile strengths, being as low as 3 or 4 per cent. The condition of hard steel often approaches that of brittleness. The fractures usually show a very fine crystalline appearance.

Besides the more usual kinds of steel having the analyses which have been indicated, there are special steel alloys which have strength properties peculiar to themselves. Among these may be mentioned nickel steel, in which a quantity of nickel is alloyed with the other constituents, manganese steel, and chrome steel.

The elastic modulus for high-carbon steel is not very much higher than for the low-carbon and medium steels, but occasionally it is found to be as high as 32,000,000 lbs. per sq. in.

Stress strain diagrams for medium and hard steels are shown on Fig. 103.

Steel Castings.—The tensile strength of steel which is run into moulds so as to form castings is not high, varying from 17 to 28 tons per sq. in. Its ductility is low and it often contains blow-holes, which militate against its strength and make it unsuitable for castings which are to have accurately machined surfaces. Steel castings are, however, much stronger than similar castings made from cast iron, and considerable advances have been made of late years by which sounder castings are being produced.

Cast Iron.—Cast iron differs greatly from the other varieties of iron which have been mentioned. It contains a larger proportion of carbon, which is partly combined and partly in a state of mechanical mixture. In the harder kinds the carbon is almost all combined, while in the softer kinds, such as are used for parts which have to be machined, it exists mostly in the form of mixed graphite. The total quantity of carbon present varies from rather less than 2 per cent. to something over 4 per cent. In addition to the carbon there is also silicon, generally from 1 to 3 per cent.; manganese from 0·2 to 2·7 per cent.; between 0·2 and 1·5 per cent. of phosphorus; and from a trace to 2·5 per cent. of sulphur.

The strength of cast iron is much greater in compression than in tension. Its tensile strength may be from 8 to 14 tons per sq. in., while the compressive strength varies from 25 to 60 tons per sq. in.

Tests of direct shear of cast iron are difficult to make and of little use, but the coefficient of torsional strength, that is, the stress as calculated from the usual torsional formula corresponding to the twisting moment which it takes to fracture a shaft, works out at from 14 to 18 tons per sq. in. in the more ordinary kinds.

There is practically no permanent set in cast iron under stress, beyond what can be measured with an extensometer. But though the permanent set is small, it takes place at very low loads, and under no stress can cast iron be called truly elastic. An approximate elastic modulus can be calculated at low stresses, and this is found to vary from 9,000,000 to 16,000,000 lbs. per sq. in.; an average value is about 15,000,000 lbs. per sq. in., or about half that of wrought iron and steel.

The most satisfactory test for cast iron is that made upon a beam of the metal loaded in the centre. The most common dimensions of such a test beam are 1 in. wide, 2 ins. deep, and loaded in the centre of a 36-in. span. A beam of this kind should withstand about 3000 lbs. with a deflection at the centre of $\frac{5}{16}$ths of an inch.

As cast iron is never perfectly elastic, the beam formula does not hold good up to the breaking load, but a value corresponding to the maximum stress from the beam formula may be calculated and called the coefficient of bending strength. It is found to be about one-and-a-half times the tensile strength.

A typical load strain diagram for cast iron is shown on Fig. 103.

Malleable Cast Iron.—If ordinary castings are treated by subjecting them at a high temperature to the oxidising influence of hæmatite ore or manganese dioxide, the carbon in the iron is reduced and the metal becomes more "steely." The resulting material is called malleable cast iron. It does not possess the characteristic brittleness of cast iron, and may be bent and hammered without fracture. Its strength is increased by the process, and is found to be from 16 to 22 tons per sq. in. in tension.

Copper.

For certain purposes the use of copper is almost indispensable. It is very ductile, an excellent conductor of heat, and has a high electrical conductivity. The strength of copper varies according to the state of the metal. The tensile strength varies from about 7 tons per sq. in. in the poorer sorts of castings to 24 tons per sq. in. in the hard drawn wire used for overhead conductors. Rolled copper, whether in plates or bars, has a maximum strength,

in the unannealed state, of about 14 tons per sq. in. When hard drawn wire or hard rolled plate or bar is softened by heating to redness and quenching in water, the tensile strength is greatly diminished, being brought down to two-thirds its former value, while its ductility is greatly increased. In the hard state the elongation is from 1 to 6 per cent., while the same metal annealed may have an elongation after fracture of 40 to 50 per cent.

Hard drawn or rolled copper should have a proportional limit

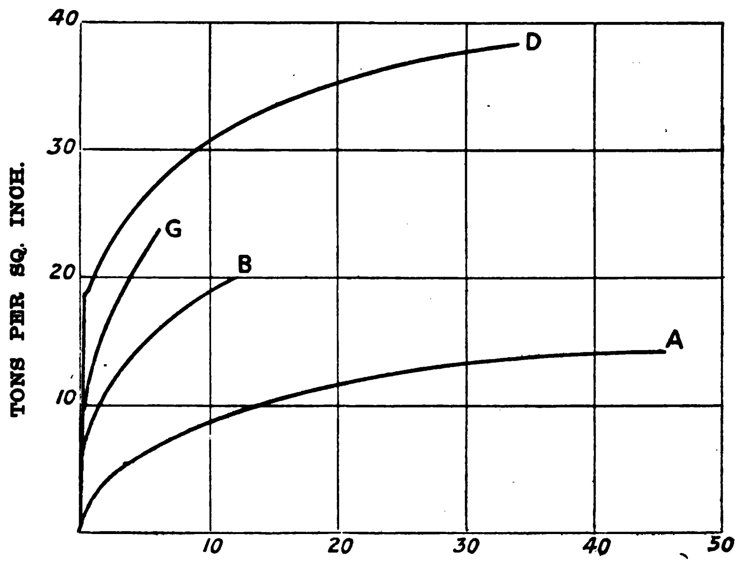

FIG. 104.—Here A refers to copper trolley wire annealed; C, the same in original state; B, gun-metal; D, aluminium bronze.

of 7 or 8 tons per sq. in.; in softened copper there is practically no limit.

The elastic modulus for copper is from 16,000,000 to 18,000,000 lbs. per sq. in.

Alloys of Copper.—The principal alloys of copper are the various bronzes containing copper and tin, along with small percentages of other metals such as zinc, aluminium, phosphorus, and manganese. The strengths of these alloys vary enormously according to the proportions of the different ingredients, and may

be from 9 tons per sq. in. in the case of the poorest quality of cast yellow brass to 40 tons per sq. in. in aluminium or manganese bronze. A good quality of gun-metal casting, such as is used for engine parts, and containing copper, say, 32 parts, tin 4, and zinc 1, should have a maximum tensile strength of 14 tons per sq. in.

Typical stress strain diagrams for copper and some of its alloys are given on Fig. 104.

Vitreous Materials.

Stone.—The strength of stone is generally specified as the number of tons per square foot of area which are required to crush it. In order to find this for any given sample, a cubical piece, from 3 ins. to 12 ins. length of side, is taken and placed between the platens of a compression testing machine. The load should be uniformly distributed over the surface, either by inserting thin layers of millboard between the platens and the specimen, or, better, by setting it in a bed of plaster of Paris.

The actual crushing strength varies from 150 tons per sq. ft. in the soft Ancaster stone, 300 for the softer kinds of sandstone, 500 for Portland limestone, 600 or 700 for the harder sandstones, 1000 for Welsh basalt, and 1500 for granite. There may be considerable variety in different samples from the same quarry.

Bricks.—Bricks should be tested for strength in the same manner as stone, great care being taken to level any inequalities exposed to pressure by previously filling up with cement or plaster of Paris.

Bricks tested in this manner yield results of which the following are a few typical examples:—

Kind of Brick.	Crushing Stress, tons per sq. ft.
London stock	140
Common wire cut, Manchester	260
Leicester wire cut	290
Accrington plastic	250
Blue Staffordshire	360
Blue Brindle, Staffordshire	480

The writer has found that the strength of bricks is from $2\frac{1}{2}$ to 5 times the strength of the brickwork in which they are used. The strength of the bricks themselves is only a very rough indication of the strength of the resulting brickwork. In order to obtain good brickwork from good bricks, it is most important that the cementing material be of the very best quality. It is useless to employ good bricks when the mortar is bad. The results of all the experiments which have been made on this point go to show that the ultimate strength of a given sample of brickwork depends very largely upon the mortar.

Cement, Mortar, and Concrete.—The most important cement used for engineering work is what is known as Portland cement. This is an artificial product, made by first mixing an earth which is composed principally of lime with one in which clay predominates. These two are ground together, wet in the right proportions, made up into lumps, dried, and then burnt like bricks. A hard material results from the calcination, and this, when ground into a fine powder, is the required cement. It consists of compound silicates and aluminates of lime. When this cement is mixed into a paste with water it sets hard, so as to form a kind of artificial stone. The time of setting varies from half an hour to several hours. In order to determine the quality of a given sample of cement, a number of tests are applied; but it is only necessary to refer here to the tests for tensile and compressive strength. The former is found by applying a tensile load to a sample made up into a suitable form for holding, and called a briquette. The tensile strength varies from 200 to 800 lbs. on the sq. in. A good cement should have a strength of 400 lbs. per sq. in. when tested at the end of a week, and should improve with age.

Crushing tests of neat cement (that is, cement alone) and mixtures of sand and cement yield results which have a more direct bearing on the suitability of a cement for engineering work, where most of the stress is compressive. They are, however, more costly to make. The compressive strength of neat cement is from 150 to 250 tons per sq. ft., while mixtures of sand and cement, such as are used for mortar, have been

shown by tests made by the writer to be somewhat as follow, for a fairly good cement:—

Ratio.	Crushing Strength, tons per sq. ft.	
Sand : Cement.	7 Weeks Old.	20 Weeks Old.
1 : 1	80	100
2 : 1	30	50
3 : 1	25	30
4 : 1	20	25
5 : 1	15	20

Portland Cement Concrete.—This is one of the most valuable materials which engineers, whether civil or mechanical, have at their disposal. By concrete is meant a compact mass, composed of small pieces of broken stone, gravel, broken brick, or other similar substance cemented together by being set in a binding matrix of cement and sand. This broken stone and sand, which is mixed with the cement, is called the aggregate or ballast; the proportion of aggregate to cement varies, a common ratio being 5 or 6 to 1. The cement and ballast are mixed dry, then sufficient water is added to produce a thin paste with the sand and cement, and the resulting mixture is tipped or shovelled into the position it has to occupy, and allowed to set.

The ultimate strength of the concrete so formed depends on the character of the aggregate employed, on the quality of the cement, on the manner and thoroughness of the mixing, on the age, and on the ratio of cement to aggregate. The writer has found that concrete made with gravel and good cement in the proportion of 5 to 1 has a crushing stress at the end of thirteen weeks of 210 tons per sq. ft., and after thirty-four weeks of 225 tons per sq. ft. Mr Deacon gives the results of tests of some samples of the concrete used on the Vyrnwy works as 107 tons per sq. ft. after one month, and 185 tons after thirty-six months. These are good results, but in some cases the strength may be as low as 50 tons per sq. ft.

The vitreous materials, including stone, brick, brickwork, cement, and concrete are all fairly elastic, and take little permanent set up to considerably above the working stresses.

On Fig. 105 is shown a stress strain diagram obtained during the loading of a brickwork pier. This is typical of almost all

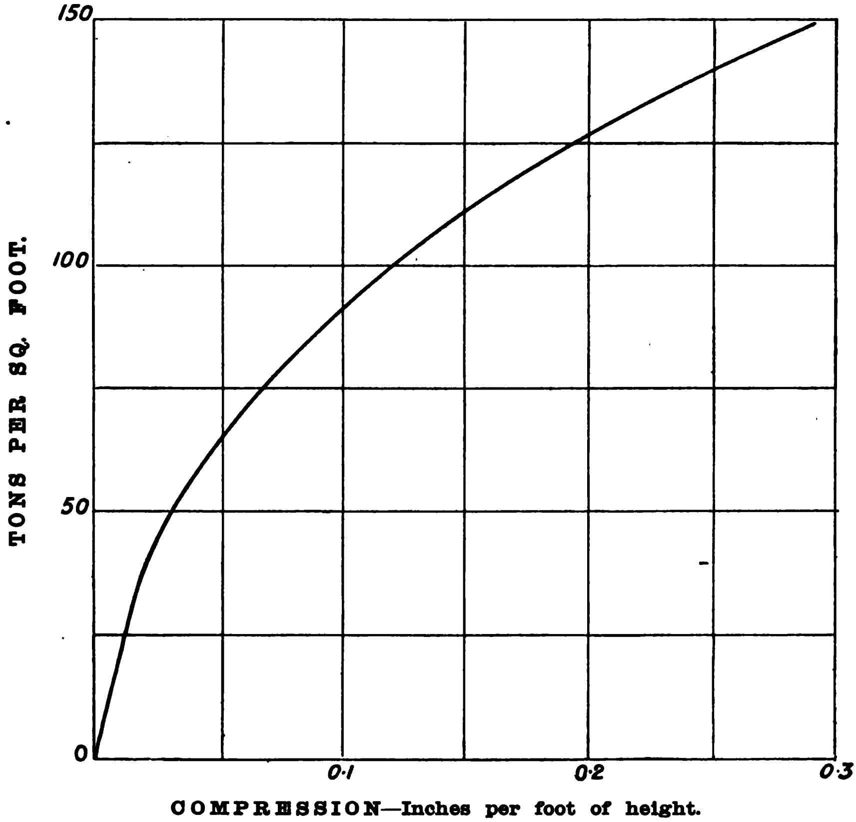

FIG. 105.

vitreous materials. The modulus of elasticity varies somewhat. The following are a few results obtained by the writer:—

Material.	Elastic Modulus—lbs. per sq. in.
Hard bricks . .	5,000,000 to 6,000,000
Softer bricks . .	1,500,000 „ 3,000,000
Good brickwork . .	1,200,000 „ 2,000,000
Poorer brickwork .	500,000 „ 1,000,000
Portland cement concrete	1,500,000 „ 2,500,000

L

Repeated and Reversed Stresses.

The effect of the repetition or reversal of the applied stresses has a most important bearing on the life of machine parts. The evidence which is available on this point is by no means complete, but sufficient has been obtained to help the designer in the proportioning of his parts. The earliest experiments on this subject were those of Wöhler and Bauschinger, and these have been supplemented more recently by Reynolds and Smith, and by Stanton at the National Physical Laboratory.

It is impossible to give anything like a complete account here of the results obtained, but a few of the more general laws arrived at may be mentioned.

Most of these experiments have been made on wrought iron and steel. The stresses were applied in different ways, and at different speeds of repetition and reversal.

Three chief methods have been adopted, namely: either the same stress ranging from zero to a certain fixed value, either in tension or compression, was applied many times in succession until rupture took place, or the successive application was from a certain stress in compression to either one of the same or a different magnitude in tension; or, again, the range was from a maximum compressive stress to a tensile stress of the same magnitude, obtained by rotating a shaft under a constant bending moment. The speed of repetition in Wöhler's tests was approximately 60 per minute. In the National Physical Laboratory experiments, the speed was intended to be more nearly that most commonly met with in modern practice, and was in the neighbourhood of 800 reversals per minute.

The main fact revealed by these experiments appears to be that the ultimate failure of a specimen depends, not so much on *the magnitude* of the stress imposed, but upon *the range* of stress. Speaking roughly, the same result would be produced by applying a stress extending from 10 tons per sq. in. in compression to 10 tons in tension, or a through range of 20 tons, as if the bar were subjected to a range of stress which reached from zero to 20 tons per sq. in. in tension.

Besides the range of stress being a factor in the ultimate failure

of the material, this was found to depend also on the number of repetitions or reversals.

With a certain range of stress it was found that the number of repetitions or reversals might be unlimited. When this range was exceeded, the number of repetitions which was needed to produce failure was smaller the greater the stress.

This limiting range has been found to nearly coincide with what has been called the *natural* elastic limit of the material, and which has already been defined. It has therefore been suggested by Unwin that it might be possible to determine the natural limit, and to take this as the safe range of stress within which any number of repetitions or reversals would not cause failure.

Unwin quotes the following results, taken from Wöhler's figures, as showing how the limiting number of reversals is affected by the range. They refer to samples of Krupp steel which were subjected to reversals of equal stress in tension and compression by rotation under a fixed bending moment.

Range of Stress.	Approximate Number of Repetitions to cause Fracture.
Tons per sq. in.	Thousands.
40·2	55
34·4	128
32·6	798
30·6	1,666
28·6	4,163
28·6	45,050

The following are three typical sets of results from the National Physical Laboratory experiments of Stanton.* The number of reversals per minute was in all the cases quoted very nearly 795 per minute. The range of stress was from a compressive stress to a tensile stress, the ratio of tension to compression being 1·4 to 1.

* *Min. Proc. Inst.C.E.*, vol. iii.

Wrought Iron, having an elastic limit of 14·28 tons per sq. in. and a maximum of 23·76 tons per sq. in.

Range of Stress. Tons per sq. in.	Approximate Number of Repetitions to cause Fracture. Thousands.	
27·86	101	
26·86	200	
25·80	253	
24·60	217	
23·43	373	
21·25	1,000	Not broken.
20·20	1,116	Not broken.
19·05	1,028	Not broken.

Mild Steel.—0·33 per cent. carbon, with 14·30 elastic limit and 28·30 maximum stress, in tons per sq. in.

Range of Stress. Tons per sq. in.	Approximate Number of Repetitions to cause Fracture. Thousands.	
30·17	68	
28·06	97	
26·4	236	
25·78	1,914	
25·71	1,330	
23·29	2,000.	Not broken.

Piston-rod Steel.—0·446 per cent. carbon, with 19·62 elastic limit and 43·85 maximum stress, in tons per sq. in.

Range of Stress. Tons per sq. in.	Approximate Number of Repetitions to cause Fracture. Thousands.	
30·16	155	
29·93	120	
28·85	308	
28·80	752	
27·88	1,032	Not broken.
27·88	3,049	Not broken.

The few figures here quoted appear to show that for every material there is a certain range of stress within which the loading may be indefinitely repeated without any perceptible weakening; but if it is extended beyond this range, fracture will eventually occur after a certain number of repetitions.

Safe Stresses Allowable in Practice.

In designing structures and machine parts, it is customary to find the stress which may be applied to any material by dividing the maximum stress as found from an ordinary tensile test by a *factor of safety*. A much more reasonable and scientific plan is to say that the material may be subjected to a certain range of stress as fixed by the results of some such experiments as those mentioned, or one which has a direct relation to the natural elastic limit.

In the light of our existing knowledge of the strength properties of the materials of construction and the effect of repetition and reversal of stress, the following may be taken as allowable working stresses.

Material.	Simple Compression.	Simple Tension.	Shear Stress in Torsion.
	Tons per sq. in.	Tons per sq. in.	Tons per sq. in.
Average Wrought Iron .	$5\frac{3}{4}$	$5\frac{3}{4}$	$3\frac{3}{8}$
Do. across grain . .	$5\frac{3}{8}$	$5\frac{3}{8}$	...
Mild Steel . . .	6	6	$3\frac{5}{8}$
Medium Steel . .	$7\frac{1}{2}$	$7\frac{1}{2}$	$5\frac{3}{8}$
High-Carbon Steel .	$9\frac{3}{8}$	$9\frac{3}{8}$	$9\frac{1}{8}$
Steel Castings . .	5	5	4
Cast Iron . . .	5	$1\frac{1}{2}$	$1\frac{1}{4}$

These are for steady loads in one direction only. Where the load is repeated so as to alternate many times between no stress and stress of one kind, the above must be multiplied by $\frac{2}{3}$; and where the stress alternates many times between tension and compression of the same amount, they must be multiplied by $\frac{1}{3}$.

The Microstructure of Metals.

In recent years much has been done to add to our knowledge of the internal structure of metals, and of the effects of stress upon this structure. A few of the main facts may be mentioned.

When a polished surface of iron or steel is placed under a microscope of sufficient power, it is seen that the surface appears to be divided up into irregular areas like a map. These areas

are the sections of the crystals which combine to form the material. It is further seen that these crystal surfaces have not all the same appearance, some being bright and others dull. The size of the grains depends upon the thermal treatment, rapid cooling leading to small crystals and slow cooling to large ones.

When the surface is examined after the application of a stress somewhat beyond the elastic limit, it is found that the grains are elongated in the direction of stress when the load is tensile. After the lapse of a considerable time at the ordinary temperature, or the application of a moderate temperature for a short time, the crystals resume their former shapes. This corresponds to what takes place when the elastic limit has been raised by stress, when it is found that the limit is again lowered to its original value after some time has elapsed or heat has been applied.

It will be remembered that fracture of materials in tension or compression generally resolves itself into failure by shear, and that this is most noticeable in the compression of brittle materials like cement and cast iron and the tensile fracture of some kinds of steel.

Professor Ewing has observed and pointed out that a somewhat similar effect is produced in the constituent grains of iron and steel when strained some little way past the proportional limit. Under a microscope of high power it is seen that each grain of material so strained is covered with rows of parallel lines, there being generally three or four systems of these inclined to one another. These lines are really minute steps representing the section of "slip planes" with the surface of the material. It thus appears that the permanent set takes place by an internal sliding, not of one grain upon another, but along parallel surfaces in the individual grains. It is by means of this slip that the change of shape of the crystals can take place.

APPENDIX

GENERAL TABLE OF STRENGTHS AND WEIGHTS

Material.	Ultimate Tensile Strength.	Crushing Strength.
IRON AND STEEL.	Tons per sq. in.	Tons per sq. in.
Wrought iron—		
Merchant bar	19 to 22	11 to 14
Ship plates	21 ,, 23	...
Best Yorkshire	23 ,, 25	12 to 16
Rivet iron	24 ,, 26	13 ,, 17
Low-carbon steel, 0·10% C.	23	13
Bridge steel, 0·20% C.	26 to 28	14 to 17
Boiler steel, 0·25% to 0·30% C.	28 ,, 32	14 ,, 18
Tyre steel, 0·28% C.	36 ,, 48	...
Medium steels, 0·30% to 0·45% C.—		
Forging steel	33 ,, 35	15 to 18
Hard forging steel	35 ,, 40	16 ,, 20
Piston-rod steel, 0·45% C.	44	20
Nickel steel	24 to 55	12 to 38
Axle steel	24 ,, 30	11 ,, 15
High-carbon steel, 0·45% to 1·50% C.—		
Manganese steel	33 ,, 42	...
Spring steel, 0·50% C.	60 ,, 70	...
Tungsten steel	70	...
Hadfield's forged cast steel	46 to 125	...
Manganese steel wire	100	...
Pianoforte wire	100 to 150	...
Ordinary steel castings	17 ,, 28	...
Hadfield's unhammered castings	32 ,, 64	...
Cast iron, very soft for machining	8 ,, 10	25 to 35
,, hard	14	60
Malleable iron castings	16 to 22	21
COPPER.		
Pure cast	14	...
Ordinary cast	6 to 10	...
Plate, hard rolled	14 ,, 15	...
Hard drawn trolley wire	24	...
Hard drawn wire, small diameter	28 to 30	...
BRONZE, ETC.		
Ordinary gunmetal castings	10	...
Good gunmetal castings	14	...
Rolled bronze, hard	27	...
Aluminium bronze	27 to 40	...
Yellow brass, good	12	...
Aluminium, rolled bar	9 to 12	...
Zinc	1 ,, 3	...
Tin	1½ ,, 2½	...
Lead	1	...

TABLE OF STRENGTHS AND WEIGHTS—*continued.*

Material.	Ultimate Tensile Strength.	Crushing Strength.
VITREOUS MATERIALS.	Tons per sq. in.	Tons per sq. ft.
Stone—		
Softer kinds of sandstone	...	150 to 300
Portland limestone	...	500
Hard sandstones	...	600 to 700
Welsh basalt	...	1000
Granite	...	1500
Brick—		
London stock	...	140
Common wire cut	...	260
Leicester wire cut	...	290
Accrington plastic	...	250
Blue Staffordshire	...	360 to 480
Brickwork—		
Common bricks and mortar	...	50
,, better mortar	...	80
Good class bricks and best mortar	...	100 to 200
Portland cement, good average	300 to 500	150
Mortar, 3 sand, 1 cement	...	30
,, 2 ,, 1 ,,	...	50
,, 1 ,, 1 ,,	...	100
Portland cement concrete, 5 to 1	...	100 to 220
TIMBER.	Breaking Strength of a 1′×1″×1″ beam. Central load, lbs.	
White pine	450	360
Memel pine	500	500
Ash	700	580
Beech	600	550
Oak	550	530
Teak	800	750
Greenheart	850	780
Pitch pine	550	570

ELASTIC OR YOUNG'S MODULUS.

Material.	Elasticity.
	Million lbs. per sq. in.
Wrought iron and steel	27 to 32
Cast iron	9 ,, 16
Copper	16 ,, 18
Gunmetal	13 ,, 16
Aluminium	10 ,, 12
Hard bricks	5 ,, 6
Softer bricks	1 ,, 3
Good brickwork	1 ,, 2
Portland cement concrete	1·5 ,, 2·5
Timber, average along the grain	1·5

WEIGHTS OF MATERIALS.

Material.	Lbs. in 1 cub. ft.
Wrought iron	485
Steel	499
Cast iron	450
Copper	552
Gunmetal	528
Brass	525
Tin	455
Zinc	437
Lead, sheet	711
Aluminium	166
,, cast	160
Sandstone	150
Portland stone	151
Welsh basalt	172
Granite	165
Bricks, soft	110
,, hard	134
Portland cement concrete	138
Lime mortar	105
Earth, average	100
White pine	30
Memel pine	35
Pitch pine	40
Beech	43
Ash	45
Spruce	32
Teak	55
Greenheart	71
Oak	53
Lignum vitæ	83

ADDITIONAL EXAMINATION QUESTIONS

Chapter I.—Elasticity.

1. Explain the meaning of the following terms:—Stress, strain, elastic limit, yield point, and modulus of elasticity.

2. Sketch typical load strain diagrams (*a*) for a tensile test of wrought iron up to the yield point, and (*b*) for a compression test of cast iron. Point out the differences in behaviour of these materials.

3. Explain the meaning of the expression "Elastic Modulus." How would you set about finding its value for a given sample of cast iron? Sketch the form of load strain diagram you would expect to obtain.

4. In a tensile test of a steel bar 1 in. in diameter, it was found that the elastic extension for each ton of load was 0·000985 in. on a length of 10 ins. Find the modulus of elasticity. [29,000,000 lbs. per sq. in.

5. A copper trolley wire 50 ft. long sustains a pull of 250 lbs., and under this load is found to be stretched 0·07 in. Find the elastic modulus, the diameter of the wire being 0·40 in. [17,050,000 lbs. per sq. in.

Chapter III.—Beams.

6. If M is the bending moment at a point in a beam, I the moment of inertia of the section at this point, f the maximum stress in the material at the section, and y the distance from the neutral axis of the section to the part where the maximum stress occurs, prove that
$$\frac{M}{I} = \frac{f}{y}$$

7. A girder resting on supports 70 ft. apart carries two rolling loads of 10 tons and 15 tons respectively. The loads roll across at a constant distance apart of 8 ft. Find the maximum bending moment on the girder.

8. What will be the stress caused in the flanges of a plate girder, 80 ft. span,

depth of $7\frac{1}{2}$ ft., and flanges 16 ins. wide, and $1\frac{3}{4}$ ins. thick, when carrying a uniform load of 1·60 tons per sq. ft. of length? [6·08 tons per sq. in.

9. The slide bars of a horizontal engine are 40 ins. long, and have a rectangular section $1\frac{1}{4}$ ins. wide. Find the depth of the bars if the greatest vertical force in the centre of a bar is 500 lbs. and the maximum stress allowable in the material is 2 tons per sq. in. [2·31 ins.

10. A cast iron beam 1·92 ins. deep, 1·03 ins. wide, when tested on a span of 36 ins., is found to break with a central load of 2760 lbs. Find what would be the central breaking load of a similar beam $3\frac{1}{2}$ ins. deep, $1\frac{3}{8}$ ins. wide, resting on supports 45 ins. apart. [9870 lbs.

11. A plate girder has a span of 70 ft. and a depth of 8 ft. The flanges are 16 ins. wide and $1\frac{1}{2}$ ins. thick. Find what load per foot run the girder must carry in order that the stress in the metal of the flanges may not exceed 6 tons per sq. in. [1·88 tons per ft.

12. What is the greatest bending moment which each point of a beam supported at the ends is liable to under a uniform load of 1 ton per ft. and a moving load of 10 tons, the span being 15 ft.? Find an expression for this, and also sketch the bending moment diagram.

13. If a beam of pitch-pine timber 1 in. wide, 1 in. deep, and 1 ft. span breaks with a central load of 800 lbs., find the safe distributed load that may be put upon a beam of the same material 15 ft. span, 4 ins. wide, and 8 ins. deep, using a factor of safety of 6. [4551 lbs.

14. Estimate the safe distributed load that can be carried by a pitch-pine beam 5 ins. wide, 9 ins. deep, and resting upon supports 18 ft. apart. [6000 lbs.

15. Find the weight of a steel girder 20 ft. long, 6 ins. deep, and having flanges 5 ins. wide. The thickness of the metal is $\frac{7}{8}$ in. in the flanges and $\frac{5}{8}$ in. in the web. [767 lbs.

16. A railway bridge 130 ft. span is covered for half its length by a train weighing 2 tons per ft. of length, one end being at the centre of the span. Find the bending moment and the shearing force at a point in the middle of the covered portion of the bridge. Also, sketch the diagrams of bending moment and shearing force. [2113 tons-feet], [32·5 tons.

17. Find the depth at the centre of a cross-girder 18 ft. span, flanges 7 ins. wide and $\frac{7}{8}$ in. thick, when the distributed load is $2\frac{1}{2}$ tons per foot run. The stress in the metal is not to exceed 6 tons per sq. in. [2 ft. 9 ins.

18. In a beam 40 ft. span, resting freely at its ends, two loads of 5 tons are placed, each 10 ft. from the centre. Find the bending moment

and shearing force at each support, at the centre, and under each load. Also, draw the bending moment and shearing force diagrams.
[At supports, M 0, S 5; at centre, M 50, S 0; at loads, M 50, S 5.

CHAPTER IV.—GRAPHICAL MOMENTS OF INERTIA.

19. Solve Example 2, Chapter IV., by calculation.

20. Prove the rule for finding graphically the modulus of the section of a rectangular beam, and point out how it is connected with the ordinary beam formula as obtained by purely mathematical reasoning.

21. The upper flange of a cast iron girder is 5 ins. wide and 1 in. thick, the lower flange is 8 ins. wide and 2 ins. thick, the total depth is 10 ins., and the web 1 in. thick. Find graphically the moment of inertia of the section. [341 inch-units.

CHAPTER V.—DEFLECTION OF BEAMS.

22. Prove that at any point in a loaded beam the relation between the radius of curvature, the bending moment, the elastic modulus of the material, and the moment of inertia of the section is $\frac{1}{R} = \frac{M}{EI}$

23. Find the central deflection of a tram rail on a clear span of 10 ft. when the elastic modulus is 29,000,000 lbs. per sq. in. The maximum stress in the material is 6 tons per sq. in. The centre of gravity of the section is 3·57 ins. from the bottom edge.
[Central load = 4·1 tons; central deflection = 0·198 in.

24. A plate girder has a span of 80 ft. and a depth of 7 ft. 6 ins. Its flanges are 16 ins. wide and 1·75 ins. thick. The stress in the metal of the flanges must not exceed 6 tons per sq. in. Find the uniformly distributed load per foot that may be put upon the girder, and the deflection in the centre under this load. The elastic modulus may be taken as 28,000,000 lbs. per sq. in. [$w = 1·57$; $\Delta = 1·025$ ins.

25. Find the load per foot run that is carried by a steel girder 30 ft. span and 18 ins. deep, the flanges being 8 ins. wide and $1\frac{1}{4}$ ins. thick, when the maximum stress in the material is 7 tons per sq. in. Also, find the deflection at the centre under this load. E = 29,000,000.
[$w = 0·809$ ton; $\Delta = 0·81$ in.

26. A road bridge over a river is carried across three spans, of which the central one is 150 ft. and the two shore spans 90 ft. each. The girders are in three portions, made up of two shore parts, each 110 ft. long, and a central part, also of 110 ft. in length. These shore por-

tions form cantilevers which project beyond the river piers, and from the ends of these the centre girder is freely hung. With a uniform load of 2 tons per ft. throughout the length of the bridge, sketch the diagrams of shearing force and bending moment.

27. A steel joist whose moment of inertia is 55·9 is tested with a central load on a span of 5 ft., and it is found that for every 8 tons increment of load there is a central deflection of 0·0506 in. Find the modulus of elasticity.

28. Prove the rule for finding the load carried by each pier of a continuous girder supported on three points equidistant and on the same level. Sketch the bending moment diagram for this case.

Chapter VI.—Shear Stress in Beams.

29. Knowing the shearing force at any section of a loaded beam and the dimensions of its section, deduce an expression which will give the intensity of the shear stress at any point in the section.

30. In Question 24 the web of the girder is $\frac{3}{8}$ in. thick. Find the mean and maximum shearing stresses in the web close to one abutment.

31. Sketch the curves showing the variation of shearing stress in the cases of—(1) A beam of rectangular section; (2) A beam of I section; (3) A tram rail.

Chapter VII.—Eccentric Loading.

32. A solid steel bar 3 ins. in diameter carries a tensile load of 40 tons. What will be the stress in the metal when the load is applied along the axis of the bar; and also, what will be the minimum and maximum stresses when the direction of the load is shifted $\frac{1}{2}$ in. from the axis?

[Tensile = 13·2 tons per sq. in.; compressive = 1·8 tons per sq. in.

33. A load of 20 tons is applied axially to a pier of brickwork $1\frac{1}{2}$ ft. square in section, at a point 6 ins. from the centre of the section measured in a direction parallel to one side. Find the maximum and minimum stresses in the material in tons per sq. ft.

[Maximum stress, 26·6; minimum stress, 8·8.

34. A rectangular bar of steel 4 ins. wide and 2 ins. thick is bent so as to form a semicircular plate, the narrow edges being curved and the wide sides remaining flat. The radius of curvature of the centre line is 5 ins. Find the tensile loads that must be applied diametri-

cally at the two ends which will cause a maximum stress in the material of 5 tons per sq. in.; also, find the minimum stress. Prove the formula.

[W = 5·33 tons; minimum stress = −4·33 tons per sq. in.

Chapter VIII.—Struts.

35. Estimate what will be the probable collapsing load of mild steel strut which is freely hinged at the ends, 15 ft. long between the hinges, and has a solid circular section 3 ins. in diameter.

Chapter IX.—Torsion.

36. If a length of shafting 3 ins. in diameter and running at 100 revolutions per minute will transmit 50 horse-power, what horse-power can be transmitted by a 5-inch shaft of the same material when running at 180 revolutions per minute? [416·6 H.P.

37. If a wrought iron shaft $3\frac{5}{8}$ ins. in diameter will safely transmit 90 horse-power at 130 revolutions per minute, find the size of steel shaft to transmit 160 horse-power at 72 revolutions, the ratio of the torsional strength of steel to that of wrought iron being as 7 to 5. [4·8 ins.

38. A solid steel shaft is to transmit 600 horse-power at a speed of 58 revolutions per minute, and the stress in the material is not to exceed 8000 lbs. per sq. in. Find the diameter. [7·46 ins.

39. What horse-power can be transmitted by a hollow shaft, whose diameters are 10 ins. and 5 ins., when running at 150 revolutions per minute, the stress in the material being limited to 8000 lbs. per sq. in.

[3500 H.P.

40. A wrought iron shaft, 4 ins. in diameter, and 30 ft. long, is used to transmit power. Find the maximum twisting moment allowable in order that the shear stress in the metal may not exceed 4 tons per sq. in., and find through how many degrees the shaft will be twisted under this moment. G = 12,000,000 lbs. per sq. in.

[T = 112,600 inch-lbs.; H.P. = 50·2; $\phi = 7·7°$.

41. A helical spring, made from round steel 0·96 in. in diameter, has an outside diameter of 4·8 ins. and consists of 10 complete coils. When unloaded, the coils nearly touch one another. Find the amount this spring will be extended under a load of $1\frac{1}{4}$ tons, when the shear modulus of the material is 11,000,000 lbs. per sq. in. [$\Delta = 1·36$ ins.

42. A hollow shaft, whose outer is twice its inner diameter, is to be used to transmit 8000 horse-power while making 76 revolutions per

minute, and the stress in the material must not exceed 7500 lbs. per sq. in. Find the outer diameter; also, find the angle of twist on a length of 50 ft. [$D_2 = 16{\cdot}9$ ins.; $\phi = 2{\cdot}1°$.

Chapter X.—Torsion and Bending.

43. In the middle of a free span of shafting, 10 ft. long between the bearings, there is a pull of 1200 lbs., at right angles to its centre line, due to a belt. The shaft is transmitting 30 horse-power at a speed of 200 revolutions per minute. The shear stress in the material must not exceed 8000 lbs. per sq. in. Find the equivalent twisting moment and the diameter. [$T_E = 73{,}200$; $M = 36{,}000$; $d = 3{\cdot}6$ ins.; $T = 9450$.

44. Prove the formula you make use of in the last question.

45. Find the shear stress in the metal of a shaft 3 ins. diameter which is transmitting 25 horse-power at a speed of 250 revolutions per minute. If there were to be a bending moment of 10,000 inch-lbs. acting on the shaft in addition to the twisting moment, what would the diameter then have to be? [1188 lbs. per sq. in.; 4·53 ins.

46. The distance from the centre of the crank pin to the centre of the near bearing, in the case of an overhung engine crank 9 ins. radius, is 10 ins. Find the diameter of the journal when the maximum pressure on the crank pin is 12,000 lbs., and the stress in the material does not exceed 7500 lbs. per sq. in. [$d = 5{\cdot}4$ ins.

Chapter XI.—Cylinders.

47. Find the thickness of a copper pipe to carry steam at a pressure of 250 lbs. per sq. in. The diameter is 7 ins., and the stress on the metal is not to exceed 2000 lbs. per sq. in. [$t = 0{\cdot}43$ in.

48. Find the diameter of a boiler steam drum in mild steel, whose diameter is to be 3 ft., and the working pressure 160 lbs. per sq. in. The stress in the material must not exceed 10,000 lbs. per sq. in. [$\frac{9}{32}$ of an inch.

Chapter XII.—Riveted Joints.

49. Two steel plates, 1 in. thick and 7 ins. wide, are allowed to overlap at their ends and are connected by three $1\frac{1}{4}$-in. steel rivets. The overlap is sufficiently great to prevent any possibility of the rivets tearing through the edges. If the rivets are made from the same steel as the plates, would you expect the joint to fail by the tearing

of the plates between the rivet holes or by the shearing of the rivets? At what load do you suppose this would take place?

[By tearing at 91 tons.

50. Design a longitudinal double-riveted lap-joint for a boiler steam drum in steel, 3 ft. diameter, 160 lbs. working pressure.

CHAPTERS XIII., XIV., AND XV.

51. Make a sketch of a typical load strain diagram as obtained during the tension test of a bar of cast iron. Define and quote values for (*a*) the maximum stress in tension and compression, and (*b*) the elastic modulus for cast iron.

52. State briefly the chief strength properties of wrought iron, steel, and cast iron, and say for what purposes each is best suited.

53. What is a load strain diagram? Sketch a typical diagram for a tensile test of wrought iron, and explain how the diagram shows the elastic limit and yield point. In what respects does cast iron differ from wrought iron as regards its elastic properties?

54. Describe the effect of repeatedly applying a tensile load which is slightly above the P-limit to a bar of steel; and how does this result differ from that obtained when the load, instead of being always tensile, is alternately tensile and compressive? State what you know as to how the results of Wöhler's experiments, and others of a similar nature, affect questions of design.

55. Sketch typical load strain diagrams (*a*) for a tensile test of wrought iron up to the yield point, and (*b*) for a compression test of cast iron. Point out the differences in behaviour of these materials.

56. What is the usual bending test for cast iron? State how it is made, and what kind of results you would expect to obtain from average material.

57. Describe the test you would employ for the purpose of ascertaining the principal tensile strength properties of a strip of mild steel plate. Make a diagrammatic sketch of a testing machine suitable for this purpose with which you are acquainted. Give the results you would expect from a good sample of structural steel, and sketch the load strain diagram.

58. A round bar of mild steel is tested in tension, with the following results:—

Original dimensions: 1·002 ins. diameter, 10 ins. long.
Final dimensions: 0·659 in. diameter, 13·06 ins. long.
Load at yield point, 13·15 tons; load, maximum, 21·90 tons.

From these data, find stress at yield point, maximum stress, percentage extension on 10 ins., and percentage reduction in area.
[16·70 tons per sq. in.; 27·80 tons per sq. in.; 30·6 per cent.; and 57 per cent.

59. Sketch the typical load strain diagram for a mild steel tension test specimen, indicating the chief points on the curve.

60. State clearly how you would set about finding out whether a given sample of Portland cement was good or bad. How would you expect the properties of two samples of the same cement to differ, one of which was coarsely ground and the other ground extremely fine?

61. In two samples of steel one is found to contain 0·2 per cent. of carbon and the other 0·5 per cent. Apart from the effects of any other constituents, what differences would you expect to find in the strength properties of these two?

62. State what you know as to the percentage of carbon in steel, and its bearing on the strength properties of different steels.

63. A cast iron beam, 1·90 ins. deep, 0·98 in. wide, when tested on a span of 36 ins., is found to break under a central load of 2500 lbs. If the corresponding dimensions had been 2 ins. and 1 in., what would you expect to have been the breaking load? State whether you consider this to be a good result, and how deflection would have been shown before fracture. Do you consider this to be a good test as compared with a tensile test, and, if so, why? [2830.

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